ON THE MODULI SPACES OF SEMI-STABLE PLANE SHEAVES OF DIMENSION ONE AND MULTIPLICITY FIVE

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Abstract. We find locally free resolutions of length one for all semi-stable sheaves supported on curves of multiplicity five in the complex projective plane. In some cases we also find geometric descriptions of these sheaves by means of extensions. We give natural stratifications for their moduli spaces and we describe the strata as certain quotients modulo linear algebraic groups. In most cases we give concrete descriptions of these quotients as fibre bundles.

1. Introduction

Let $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ denote the moduli space of semi-stable sheaves $\mathcal{F}$ on the complex projective plane $\mathbb{P}^2$ with support of dimension 1, multiplicity r and Euler characteristic χ. The Hilbert polynomial of $\mathcal{F}$ is $P_{\mathcal{F}}(t) = rt + \chi$ and the ratio $p(\mathcal{F}) = \chi/r$ is the slope of $\mathcal{F}$. We recall that $\mathcal{F}$ is semi-stable, respectively stable, if $\mathcal{F}$ is pure (meaning that there are no proper subsheaves with support of dimension zero) and any proper subsheaf $\mathcal{F}' \subset \mathcal{F}$ satisfies $p(\mathcal{F}') \le p(\mathcal{F})$, respectively $p(\mathcal{F}') < p(\mathcal{F})$. The spaces $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ for $r \le 3$ are completely understood from the work of Le Potier [8], and others. In [4] Drézet and the author studied the spaces $\mathrm{M}_{\mathbb{P}^2}(4,\chi)$. This paper is concerned with the geometry of the spaces $\mathrm{M}_{\mathbb{P}^2}(5,\chi)$. In view of the obvious isomorphism $\mathrm{M}_{\mathbb{P}^2}(r,\chi) \simeq \mathrm{M}_{\mathbb{P}^2}(r,\chi+r)$ sending the stable-equivalence class of a sheaf $\mathcal{F}$ to the stable-equivalence class of the twisted sheaf $\mathcal{F} \otimes \mathcal{O}(1)$, it is enough to assume that $0 \le \chi \le 4$. According to [8], the spaces $\mathrm{M}_{\mathbb{P}^2}(5,\chi)$ are projective, irreducible, locally factorial, of dimension 26 and smooth at all points given by stable sheaves. In particular, $\mathrm{M}_{\mathbb{P}^2}(5,\chi)$, $1 \le \chi \le 4$, are smooth.

In this paper we shall carry out the same program as in [4]. We shall decompose each moduli space into locally closed subvarieties, called *strata*, by means of cohomological conditions. Given a stratum $X \subset \mathrm{M}_{\mathbb{P}^2}(5,\chi)$, we shall find locally free sheaves $\mathcal{A}$ and $\mathcal{B}$ on $\mathbb{P}^2$ such that each sheaf $\mathcal{F}$ giving a point in X admits a presentation

$$0 \longrightarrow \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \longrightarrow \mathcal{F} \longrightarrow 0.$$

1991 *Mathematics Subject Classification.* Primary 14D20, 14D22.

The linear algebraic group $G = (\mathrm{Aut}(\mathcal{A}) \times \mathrm{Aut}(\mathcal{B}))/\mathbb{C}^*$ acts by conjugation on the finite dimensional vector space $\mathbb{W} = \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Here $\mathbb{C}^*$ is embedded as the subgroup of homotheties. The set of morphisms φ appearing above is a locally closed subset $W \subset \mathbb{W}$, which is invariant under the action of G. We shall prove that a good or a categorical quotient of W by G exists and is isomorphic to X. The existence of the good quotient does not follow from the geometric invariant theory if G is non-reductive, which, most of the time, will be our case. In some cases we shall describe the sheaves in the strata by means of extensions.

Throughout this paper we keep the notations and conventions from [4]. We work over the complex numbers. We fix a vector space V over $\mathbb{C}$ of dimension 3 and we identify $\mathbb{P}^2$ with the space $\mathbb{P}(V)$ of lines in V. We fix a basis $\{X, Y, Z\}$ of V^*. If $\mathcal{A}$ and $\mathcal{B}$ are direct sums of line bundles on $\mathbb{P}^2$, we identify $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ with the space of matrices with entries in appropriate symmetric powers of V^*, i.e. matrices with entries homogeneous polynomials in X, Y, Z. We especially refer to the section of preliminaries in [4], which contains most of the techniques that we shall use.

According to [10], there is a duality isomorphism $\mathrm{M}_{\mathbb{P}^2}(r, \chi) \simeq \mathrm{M}_{\mathbb{P}^2}(r, -\chi)$ sending the stable-equivalence class of a sheaf $\mathcal{F}$ to the stable-equivalence class of the dual sheaf $\mathcal{F}^{\mathrm{D}} = \mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}^2})$. This allows us to study the spaces $\mathrm{M}_{\mathbb{P}^2}(5, \chi)$ in pairs. Thus $\mathrm{M}_{\mathbb{P}^2}(5, 3)$ and $\mathrm{M}_{\mathbb{P}^2}(5, 2)$ are isomorphic and will be studied in section 2. The spaces $\mathrm{M}_{\mathbb{P}^2}(5, 1)$ and $\mathrm{M}_{\mathbb{P}^2}(5, 4)$ are, likewise, isomorphic and will be treated in section 3. The last section deals with $\mathrm{M}_{\mathbb{P}^2}(5, 0)$. In the remaining part of this introduction we shall make a summary of results.

1.1. **The moduli spaces $\mathrm{M}_{\mathbb{P}^2}(5, 3)$ and $\mathrm{M}_{\mathbb{P}^2}(5, 2)$.** We shall decompose the moduli space $\mathrm{M}_{\mathbb{P}^2}(5, 3)$ into four strata: an open stratum X_0, two locally closed strata X_1, X_2 and a closed stratum X_3. X_1 is a proper open subset inside a fibre bundle over $\mathbb{P}^2 \times \mathrm{N}(3, 2, 3)$ with fibre $\mathbb{P}^{16}$. Here $\mathrm{N}(3, 2, 3)$ is the moduli space of semi-stable Kronecker modules $\tau \colon \mathbb{C}^2 \otimes V \to \mathbb{C}^3$. X_2 is a proper open subset inside a fibre bundle over $\mathrm{N}(3, 3, 2)$ with fibre $\mathbb{P}^{17}$. X_3 is isomorphic to the Hilbert flag scheme of quintic curves in $\mathbb{P}^2$ containing zero-dimensional subschemes of length 2.

A sheaf $\mathcal{F}$ from $\mathrm{M}_{\mathbb{P}^2}(5, 3)$ gives a point in X_0 if and only if the following cohomological conditions are satisfied:

$$h^0(\mathcal{F}(-1)) = 0, \qquad h^1(\mathcal{F}) = 0, \qquad h^0(\mathcal{F} \otimes \Omega^1(1)) = 1.$$

Each semi-stable sheaf whose stable-equivalence class is in X_0 has a resolution of the form

$$0 \longrightarrow 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

We consider the vector space $\mathbb{W} = \mathrm{Hom}(2\mathcal{O}(-2) \oplus \mathcal{O}(-1), 3\mathcal{O})$ and the linear algebraic group

$$G = (\mathrm{Aut}(2\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(3\mathcal{O}))/\mathbb{C}^*$$

acting on $\mathbb{W}$ by conjugation. $\mathbb{C}^*$ is embedded as the subgroup of homotheties here. The set of morphisms φ occuring above forms an open G-invariant subset $W \subset \mathbb{W}$ given by the following conditions: φ is injective and φ is not in the orbit of a morphism represented by a matrix of the form

$$\left[\begin{array}{ccc} \star & \star & \star \\ \star & \star & 0 \\ \star & \star & 0 \end{array}\right] \qquad \text{or} \qquad \left[\begin{array}{ccc} \star & \star & \star \\ \star & \star & \star \\ \star & 0 & 0 \end{array}\right].$$

W admits a geometric quotient W/G modulo G and $W/G \simeq X_0$. The information about X_0 is summarised in the second row of the following table. The other rows of the table contain the analogous information about the remaining strata of $\mathrm{M}_{\mathbb{P}^2}(5,3)$. The last column gives the codimension of each stratum. For each W there is a geometric quotient W/G modulo the canonical group G acting by conjugation on the ambient vector space $\mathbb{W}$ of homomorphisms of sheaves and W/G is isomorphic to the corresponding stratum of $\mathrm{M}_{\mathbb{P}^2}(5,3)$.

Table 1. Summary for $\mathrm{M}_{\mathbb{P}^2}(5,3)$.

stratum	cohomological conditions	subset $W \subset \mathbb{W}$ of morphisms φ	codim.
X_0	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$	$2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O}$ φ is injective φ is not equivalent to $\left[\begin{array}{ccc} \star & \star & \star \\ \star & \star & 0 \\ \star & \star & 0 \end{array}\right]$ or $\left[\begin{array}{ccc} \star & \star & \star \\ \star & \star & \star \\ \star & 0 & 0 \end{array}\right]$	0
X_1	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 2$	$2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 3\mathcal{O}$ φ is injective $\varphi_{12} = 0$ φ_{11} has linearly independent entries φ_{22} has linearly independent maximal minors	2
X_2	$h^0(\mathcal{F}(-1)) = 1$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 3$	$3\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O}(1)$ φ is injective φ_{11} has linearly independent maximal minors	3
X_3	$h^0(\mathcal{F}(-1)) = 1$ $h^1(\mathcal{F}) = 1$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 4$	$\mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O} \oplus \mathcal{O}(1)$ φ is injective $\varphi_{12} \neq 0$ $\varphi_{12} \nmid \varphi_{22}$	4

Applying to X_i the duality isomorphism $\mathrm{M}_{\mathbb{P}^2}(5,3) \to \mathrm{M}_{\mathbb{P}^2}(5,2)$ of [10] defined by

$$\mathcal{F} \longrightarrow \mathcal{F}^{\mathrm{D}}(1) = \mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}^2}) \otimes \mathcal{O}(1),$$

we get a dual stratum $X_i^{\mathrm{D}} \subset \mathrm{M}_{\mathbb{P}^2}(5,2)$ given by the cohomological conditions derived from Serre duality (see 2.1.2 [4]). For instance, X_0^{D} consists of those sheaves $\mathcal{G}$ in $\mathrm{M}_{\mathbb{P}^2}(5,2)$ satisfying the conditions

$$h^1(\mathcal{G}) = 0, \qquad h^0(\mathcal{G}(-1)) = 0, \qquad h^1(\mathcal{G} \otimes \Omega^1(1)) = 1.$$

According to [10], lemma 3, taking the dual of each term in a locally free resolution of length 1 for $\mathcal{F}$ gives a resolution for $\mathcal{F}^{\mathrm{D}}$. Thus every sheaf $\mathcal{G}$ in X_0^{D} has a resolution of the form

$$0 \longrightarrow 3\mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

The conditions on ψ are the transposed conditions on the morphism φ from above. In this fashion we get a "dual table" for $\mathrm{M}_{\mathbb{P}^2}(5,2)$. We omit the details.

Inside X_0 there is an open dense subset of sheaves that have a presentation of the form

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow \Omega^1(2) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The complement in X_0 of this subset, denoted X_{01}, has codimension 1. The generic sheaves giving points in X_{01} have the form

$$\mathcal{O}_C(1)(P_1 + P_2 + P_3 + P_4 - P_5),$$

where $C \subset \mathbb{P}^2$ is a smooth quintic curve, $P_1, \ldots, P_5$ are distinct points on C and P_1, P_2, P_3, P_4 are in general linear position.

The sheaves giving points in X_1 are precisely the non-split extension sheaves of the form

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathcal{G}$ varies in X_2^{D} and $\mathbb{C}_x$ is the structure sheaf of a closed point in the support of $\mathcal{G}$.

The sheaves $\mathcal{G}$ in X_2^{D} are either of the form $\mathcal{J}_Z(2)$, where $\mathcal{J}_Z \subset \mathcal{O}_C$ is the ideal sheaf of a zero-dimensional subscheme of length 3 contained in a quintic curve C, Z not contained in a line, or they are extension sheaves of the form

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_D(1) \longrightarrow 0,$$

where L is a line and D is a quartic curve, that are not in the kernel of the canonical map

$$\mathrm{Ext}^1(\mathcal{O}_D(1), \mathcal{O}_L(-1)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}_L(-1)).$$

The sheaves in X_3 are the twisted ideal sheaves $\mathcal{J}_Z(2) \subset \mathcal{O}_C(2)$ of zero-dimensional subschemes Z of length 2 contained in quintic curves C.

1.2. **The moduli spaces** $\mathrm{M}_{\mathbb{P}^2}(5,1)$ **and** $\mathrm{M}_{\mathbb{P}^2}(5,4)$**.** We shall decompose the moduli space $\mathrm{M}_{\mathbb{P}^2}(5,1)$ into four strata: an open stratum X_0, two locally closed strata X_1, X_2 and a closed stratum X_3. X_0 is a proper open subset inside a fibre bundle with base $\mathrm{N}(3,4,3)$ and fibre $\mathbb{P}^{14}$. X_1 is a proper open subset inside a fibre bundle with base $\mathrm{Grass}(2,\mathbb{C}^6)$ and fibre $\mathbb{P}^{16}$. X_2 is a proper open subset inside a fibre bundle with fibre $\mathbb{P}^{17}$ and base $Y \times \mathbb{P}^2$, where Y is the Hilbert scheme of zero-dimensional subschemes of $\mathbb{P}^2$ of length 2. X_3 is the universal quintic in $\mathbb{P}^2 \times \mathbb{P}(S^5 V^*)$.

The information about the cohomological conditions defining each stratum in $\mathrm{M}_{\mathbb{P}^2}(5,1)$ and resolutions for semi-stable sheaves can be found in table 2 below. This is organised as table 1, so we refer to the previous subsection for the meaning of the different items. Again, each X_i is isomorphic to the corresponding geometric quotient W/G. By duality, from table 2 can be obtained a table for $\mathrm{M}_{\mathbb{P}^2}(5,4)$, which we do not include here.

Table 2. Summary for $\mathrm{M}_{\mathbb{P}^2}(5,1)$.

	cohomological conditions	subset $W \subset \mathbb{W}$ of morphisms φ	
X_0	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F}\otimes\Omega^1(1)) = 0$	$4\mathcal{O}(-2) \xrightarrow{\varphi} 3\mathcal{O}(-1) \oplus \mathcal{O}$ φ is injective φ_{11} is not equivalent to $\begin{bmatrix} \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & 0 \end{bmatrix}$ or $\begin{bmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \end{bmatrix}$ or $\begin{bmatrix} \star & 0 & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}$	0
X_1	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 1$ $h^0(\mathcal{F}\otimes\Omega^1(1)) = 0$	$\mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}$ φ is injective φ_{12} and φ_{22} are linearly independent two-forms	2
X_2	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 1$ $h^0(\mathcal{F}\otimes\Omega^1(1)) = 1$	$\mathcal{O}(-3)\oplus\mathcal{O}(-2)\oplus\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1)\oplus 2\mathcal{O}$ φ is injective $\varphi_{13} = 0$ $\varphi_{12} \neq 0$, $\varphi_{12} \nmid \varphi_{11}$ φ_{23} has linearly independent entries	3
X_3	$h^0(\mathcal{F}(-1)) = 1$ $h^1(\mathcal{F}) = 2$ $h^0(\mathcal{F}\otimes\Omega^1(1)) = 3$	$2\mathcal{O}(-3) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O}(1)$ φ is injective φ_{11} has linearly independent entries	5

Inside X_0 there is an open dense subset consisting of sheaves of the form $\mathcal{J}_Z(2)^{\mathrm{D}}$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 6 not contained in a conic curve, contained in a quintic curve C, and $\mathcal{J}_Z \subset \mathcal{O}_C$ is its ideal

sheaf. The complement in X_0 of this open subset is the disjoint union of two sets X_{01} and X_{02}. The sheaves in X_{01} occur as non-split extensions of one of the following three kinds:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_Z(1)^{\mathrm{D}} \longrightarrow 0.$$

Here $L \subset \mathbb{P}^2$ is a line, $\mathcal{G}$ is in the exceptional divisor of $\mathrm{M}_{\mathbb{P}^2}(4,0)$, $\mathcal{E}$ is the twist by -1 of a sheaf in the stratum $X_3 \subset \mathrm{M}_{\mathbb{P}^2}(5,3)$, $Y \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 3 not contained in a line, contained in the support of $\mathcal{E}$, $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 3 not contained in a line, contained in a quartic curve C, and $\mathcal{J}_Z \subset \mathcal{O}_C$ is its ideal sheaf. Not all of the above extension sheaves are in X_{01}, namely there are certain conditions that must be satisfied for which we refer to subsection 3.3. For X_{02} we can be more specific. A sheaf $\mathcal{F}$ gives a point in X_{02} precisely if it is an extension of the form

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

and satisfies $\mathrm{H}^1(\mathcal{F}) = 0$. Here C' is a cubic curve, C is a conic curve in $\mathbb{P}^2$.

The sheaves $\mathcal{F}$ in X_1 are either of the form $\mathcal{J}_Z(2)$, where $Z \subset \mathbb{P}^2$ is the intersection of two conic curves without common component, Z is contained in a quintic curve C and $\mathcal{J}_Z \subset \mathcal{O}_C$ is its ideal sheaf, or they are extension sheaves of the form

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_x(1) \longrightarrow 0,$$

satisfying the relation $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$. Here $L \subset \mathbb{P}^2$ is a line and $\mathcal{J}_x \subset \mathcal{O}_{C'}$ is the ideal sheaf of a closed point x on a quartic curve $C' \subset \mathbb{P}^2$.

The generic sheaves from X_2 are of the form $\mathcal{O}_C(1)(-P_1 + P_2 + P_3)$, where $C \subset \mathbb{P}^2$ is a smooth quintic curve and P_1, P_2, P_3 are distinct points on C.

The sheaves giving points in X_3 are precisely the non-split extension sheaves of the form

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

Here $C \subset \mathbb{P}^2$ is a quintic curve and $\mathbb{C}_x$ is the structure sheaf of a closed point.

1.3. **The moduli space** $\mathrm{M}_{\mathbb{P}^2}(5,0)$**.** This moduli space can be decomposed into four strata: an open stratum X_0, two locally closed strata X_1, X_2 and a closed stratum X_3. X_0 is a proper open subset inside $\mathrm{N}(3,5,5)$. X_2 is a proper open subset inside a fibre bundle over $\mathbb{P}^2 \times \mathbb{P}^2$ with fibre $\mathbb{P}^{18}$. X_3 consists of sheaves of the form $\mathcal{O}_C(1)$, where $C \subset \mathbb{P}^2$ is a quintic curve, and is isomorphic to $\mathbb{P}(S^5 V^*)$. All strata are invariant under the duality isomorphism.

The information about the cohomological conditions defining each stratum and resolutions for semi-stable sheaves can be found in table 3 below, which is organised as table 1. X_0 is a good quotient, X_1 is a categorical quotient and X_2 is a geometric quotient of W by G.

Table 3. Summary for $\mathrm{M}_{\mathbb{P}^2}(5,0)$.

stratum	cohomological conditions	subset $W \subset \mathbb{W}$ of morphism φ	codim.
X_0	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$	$5\mathcal{O}(-2) \xrightarrow{\varphi} 5\mathcal{O}(-1)$ φ is injective	0
X_1	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 1$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$	$\mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O}$ φ is injective φ_{12} is injective	1
X_2	$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 2$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$	$2\mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus 2\mathcal{O}$ φ is injective φ_{11} has linearly independent entries φ_{22} has linearly independent entries	4
X_3	$h^0(\mathcal{F}(-1)) = 1$ $h^1(\mathcal{F}) = 3$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 3$	$\mathcal{O}(-4) \xrightarrow{\varphi} \mathcal{O}(1)$ $\varphi \neq 0$	6

The generic sheaves in X_0 are of the form $\mathcal{J}_Z(3)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 10 not contained in a cubic curve, contained in a quintic curve C, and $\mathcal{J}_Z \subset \mathcal{O}_C$ is its ideal sheaf.

The sheaves giving points in X_2 are precisely the non-split extension sheaves of the form

$$0 \longrightarrow \mathcal{J}_x(1) \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_z \longrightarrow 0,$$

where $\mathcal{J}_x \subset \mathcal{O}_C$ is the ideal sheaf of a closed point x on a quintic curve $C \subset \mathbb{P}^2$ and $\mathbb{C}_z$ is the structure sheaf of a closed point $z \in C$. When $x = z$ we exclude the possibility $\mathcal{F} \simeq \mathcal{O}_C(1)$.

Acknowledgements

The author was supported by the BitDefender Scholarship at the Institute of Mathematics of the Romanian Academy and by the Consiliul Naţional al Cercetării Ştiinţifice, grant PN II–RU 169/2010 PD–219. The author is grateful to J.-M. Drézet for many insightful remarks and helpful suggestions. Important changes were made after the referee report. The referee suggested improvements to the proofs of 2.1.4 and 2.1.6, noticed gaps in our original construction of quotients as fibre bundles, and pointed out the argument used to show that the generic sheaves at 2.3.2 and 3.3.4 are certain line bundles supported on smooth quintics

2. Euler characteristic two or three

2.1. **Locally free resolutions for semi-stable sheaves.**

Proposition 2.1.1. *There are no sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(5,3)$ and satisfying the conditions $h^0(\mathcal{F}(-1)) = 0$ and $h^1(\mathcal{F}) \neq 0$.*

Proof. According to 6.4 [9], there are no sheaves $\mathcal{G}$ in $\mathrm{M}_{\mathbb{P}^2}(5,2)$ satisfying the conditions $h^0(\mathcal{G}(-1)) \neq 0$ and $h^1(\mathcal{G}) = 0$. The result follows by duality. $\square$

From this and from 4.3 [9] we obtain:

Proposition 2.1.2. *Let $\mathcal{F}$ be a sheaf in $\mathrm{M}_{\mathbb{P}^2}(5,3)$ satisfying the condition $h^0(\mathcal{F}(-1)) = 0$. Then $h^1(\mathcal{F}) = 0$ and $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$ or 2. The sheaves from the first case are precisely the sheaves that have a resolution of the form*

(i) $$0 \longrightarrow 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

with φ not equivalent, modulo the action of the natural group of automorphisms, to a morphism represented by a matrix of the form

$$\left[\begin{array}{ccc} \star & \star & \star \\ \star & \star & 0 \\ \star & \star & 0 \end{array}\right] \qquad \text{or} \qquad \left[\begin{array}{ccc} \star & \star & \star \\ \star & \star & \star \\ \star & 0 & 0 \end{array}\right].$$

The sheaves in the second case are precisely the sheaves that have a resolution of the form

(ii) $$0 \longrightarrow 2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \left[\begin{array}{cc} \varphi_{11} & 0 \\ \varphi_{21} & \varphi_{22} \end{array}\right], \qquad \varphi_{11} = \left[\begin{array}{cc} \ell_1 & \ell_2 \end{array}\right],$$

where ℓ_1, ℓ_2 are linearly independent one-forms and the maximal minors of φ_{22} are linearly independent two-forms.

Proposition 2.1.3. *Let $\mathcal{F}$ be a sheaf giving a point in $\mathrm{M}_{\mathbb{P}^2}(5,3)$ and satisfying the conditions $h^1(\mathcal{F}) = 0$ and $h^0(\mathcal{F}(-1)) \neq 0$. Then $h^0(\mathcal{F}(-1)) = 1$. These sheaves are precisely the sheaves with resolution of the form*

$$0 \longrightarrow 3\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{11} has linearly independent maximal minors.

Proof. The first conclusion follows from 6.6 [9]. According to 5.3 [9], every sheaf $\mathcal{G}$ in $\mathrm{M}_{\mathbb{P}^2}(5,2)$ satisfying $h^0(\mathcal{G}(-1)) = 0$ and $h^1(\mathcal{G}) = 1$ has a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \xrightarrow{\psi} 3\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

in which ψ_{12} has linearly independent maximal minors. The second conclusion follows by duality. $\square$

Proposition 2.1.4. *Let $\mathcal{F}$ be a sheaf giving a point in $\mathrm{M}_{\mathbb{P}^2}(5,3)$ and satisfying the conditions $h^0(\mathcal{F}(-1)) = 1$ and $h^1(\mathcal{F}) = 1$. Then $h^0(\mathcal{F} \otimes \Omega^1(1)) = 4$ and $\mathcal{F}$ has a resolution of the form*

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0$$

with $\varphi_{12} \neq 0$ and φ_{22} not divisible by φ_{12}. Conversely, every $\mathcal{F}$ having such a resolution is semi-stable.

Proof. Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(5,3)$ and satisfy the cohomological conditions from the proposition. Write $m = h^0(\mathcal{F} \otimes \Omega^1(1))$. The Beilinson free monad (2.2.1) [4] for $\mathcal{F}$ reads

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow 3\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow (m-1)\mathcal{O}(-1) \oplus 4\mathcal{O} \longrightarrow \mathcal{O} \longrightarrow 0$$

and gives the resolution

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow 3\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow \Omega^1 \oplus (m-4)\mathcal{O}(-1) \oplus 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

We see from the above that $m \geq 4$. Combining with the Euler sequence we obtain the resolution

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3) \oplus 3\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} \\ 3\mathcal{O}(-2) \oplus (m-4)\mathcal{O}(-1) \oplus 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0, \end{aligned}$$

$$\psi = \begin{bmatrix} 0 \\ 0 \\ \psi_{31} \end{bmatrix}, \qquad \varphi = \begin{bmatrix} \eta & \varphi_{12} & 0 \\ 0 & \varphi_{22} & 0 \\ 0 & \varphi_{32} & \varphi_{33} \end{bmatrix}. \qquad \text{Here} \qquad \eta = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

We have a commutative diagram in which the vertical maps are projections onto direct summands:

$$\begin{tikzcd} \mathcal{O}(-3) \oplus 3\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \arrow[r, "\varphi"] \arrow[d] & 3\mathcal{O}(-2) \oplus (m-4)\mathcal{O}(-1) \oplus 4\mathcal{O} \arrow[d] \\ \mathcal{O}(-3) \oplus 3\mathcal{O}(-2) \arrow[r, "\alpha"] & 3\mathcal{O}(-2) \end{tikzcd},$$

$$\alpha = \begin{bmatrix} \eta & \varphi_{12} \end{bmatrix}.$$

Thus $\mathcal{F}$ maps surjectively to $\mathcal{C}oker(\alpha)$. If $\operatorname{rank}(\varphi_{12}) = 0$, then $\mathcal{C}oker(\alpha) \simeq \Omega^1$. If $\operatorname{rank}(\varphi_{12}) = 1$, then $\mathcal{C}oker(\alpha) \simeq \mathcal{I}_x(-1)$, where $\mathcal{I}_x \subset \mathcal{O}$ is the ideal sheaf of a point $x \in \mathbb{P}^2$. These two cases are unfeasible because $\mathcal{F}$ has support of dimension 1 so it cannot map surjectively onto a sheaf supported on the entire plane. If $\operatorname{rank}(\varphi_{12}) = 2$, then $\mathcal{C}oker(\alpha)$ would be isomorphic to $\mathcal{O}_L(-2)$ for a line $L \subset \mathbb{P}^2$, so it would destabilise $\mathcal{F}$. We conclude that $\operatorname{rank}(\varphi_{12}) = 3$. We may cancel $3\mathcal{O}(-2)$ to get the resolution

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} (m-4)\mathcal{O}(-1) \oplus 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\psi = \begin{bmatrix} 0 \\ \psi_{21} \end{bmatrix}, \qquad \varphi = \begin{bmatrix} \varphi_{11} & 0 \\ \varphi_{21} & \varphi_{22} \end{bmatrix}.$$

Note that $\mathcal{F}$ maps surjectively onto $\mathcal{C}oker(\varphi_{11})$, so the latter has rank zero, forcing $m \leq 5$. If $m = 5$, then $\mathcal{C}oker(\varphi_{11})$ would be isomorphic to $\mathcal{O}_C(-1)$

for a conic curve $C \subset \mathbb{P}^2$, so it would destabilise $\mathcal{F}$. We deduce that $m = 4$ and we get the resolution

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3) \oplus 4\mathcal{O}(-1) \xrightarrow{\varphi} 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Let $\bar{\psi}\colon V \to \mathbb{C}^4$ be the linear map induced by ψ_{21}. Let H be the image of $\bar{\psi}$ and let $K \subset \mathbb{C}^4$ be a linear subspace such that $H \oplus K = \mathbb{C}^4$. We have an exact sequence

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3) \oplus (K \otimes \mathcal{O}(-1)) \oplus (H \otimes \mathcal{O}(-1)) \xrightarrow{\varphi} 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

in which $\psi_{11} = 0$, $\psi_{21} = 0$. If $\dim(H) = 1$, then ψ_{31} is generically surjective. As φ vanishes on $\mathcal{I}m(\psi_{31})$, it must vanish on $H \otimes \mathcal{O}(-1)$, hence $H \otimes \mathcal{O}(-1)$ is a subsheaf of $\mathcal{O}(-2)$. This is absurd. If $\dim(H) = 2$, then $\mathcal{C}oker(\psi_{31})$ is isomorphic to the ideal sheaf $\mathcal{I}_x$ of a point $x \in \mathbb{P}^2$. We get a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \oplus \mathcal{I}_x \longrightarrow 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

The image of $\mathcal{I}_x$ is included into a factor $\mathcal{O}$ of $4\mathcal{O}$ because $\mathrm{Hom}(\mathcal{I}_x, \mathcal{O}) \simeq \mathbb{C}$. We obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_x \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & \mathbb{C}_x \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \oplus \mathcal{I}_x \arrow[r] & 4\mathcal{O} \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

in which the first two vertical maps are injective. The induced map $\mathbb{C}_x \to \mathcal{F}$ is zero because $\mathcal{F}$ has no zero-dimensional torsion. It follows that $\mathcal{O}$ is a subsheaf of $\mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \oplus \mathcal{I}_x$, which is absurd. We deduce that H has dimension 3, so $\mathcal{C}oker(\psi_{31}) \simeq \Omega^1(1)$ and we get the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1) \oplus \Omega^1(1) \xrightarrow{\varphi} 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Consider the canonical morphism $i\colon \Omega^1(1) \to \mathrm{Hom}(\Omega^1(1), \mathcal{O})^* \otimes \mathcal{O} \simeq 3\mathcal{O}$. There is a morphism $\beta\colon 3\mathcal{O} \to 4\mathcal{O}$ such that $\beta \circ i = \varphi_{13}$. If β were not injective, then φ would be equivalent to a morphism represented by a matrix of the form

$$\left[\begin{array}{cc} \gamma_{11} & 0 \\ \gamma_{21} & \gamma_{22} \end{array} \right],$$

where $\gamma_{11} \in \mathrm{Hom}(\mathcal{O}(-3) \oplus \mathcal{O}(-1), 2\mathcal{O})$. But then $\mathcal{C}oker(\gamma_{11})$ would be a destabilising quotient sheaf of $\mathcal{F}$. Thus β is injective, from which we deduce that $\mathcal{C}oker(\varphi_{13}) \simeq \mathcal{O} \oplus \mathcal{O}(1)$. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

If $\varphi_{12} = 0$, then $\mathcal{F}$ would have a destabilising subsheaf of the form $\mathcal{O}_C(1)$, for a conic curve $C \subset \mathbb{P}^2$. If φ_{12} divided φ_{22}, then $\mathcal{F}$ would have a destabilising subsheaf of the form $\mathcal{O}_L$ for a line $L \subset \mathbb{P}^2$.

Conversely, assume that $\mathcal{F}$ has a resolution as in the proposition. Then $\mathcal{F}$ has no zero-dimensional torsion because it has projective dimension 1 at every point in its support. Thus, it is enough to show that $\mathcal{F}$ cannot have a destabilising subsheaf. Let $\mathcal{F}' \subset \mathcal{F}$ be a non-zero subsheaf of multiplicity at most 4. According to 2.3.5, $\mathcal{F}$ is isomorphic to $\mathcal{J}_Z(2)$, where $\mathcal{J}_Z \subset \mathcal{O}_C$ is the ideal sheaf of a zero-dimensional scheme Z of length 2 inside a quintic curve C. According to [9], lemma 6.7, there is a sheaf $\mathcal{A} \subset \mathcal{O}_C(2)$ containing $\mathcal{F}'$ such that $\mathcal{A}/\mathcal{F}'$ is supported on finitely many points and $\mathcal{O}_C(2)/\mathcal{A} \simeq \mathcal{O}_S(2)$ for a curve $S \subset C$ of degree $d \leq 4$. The slope of $\mathcal{F}'$ can be estimated as follows:

$$\begin{aligned} P_{\mathcal{F}'}(t) &= P_{\mathcal{A}}(t) - h^0(\mathcal{A}/\mathcal{F}') \\ &= P_{\mathcal{O}_C}(t+2) - P_{\mathcal{O}_S}(t+2) - h^0(\mathcal{A}/\mathcal{F}') \\ &= (5-d)t + \frac{(d-5)(d-2)}{2} - h^0(\mathcal{A}/\mathcal{F}'), \\ p(\mathcal{F}') &= \frac{2-d}{2} - \frac{h^0(\mathcal{A}/\mathcal{F}')}{5-d} \leq \frac{1}{2} < \frac{3}{5} = p(\mathcal{F}). \end{aligned}$$

We conclude that $\mathcal{F}$ is semi-stable. □

Proposition 2.1.5. *Any sheaf $\mathcal{G}$ giving a point in $\mathrm{M}_{\mathbb{P}^2}(5,2)$ satisfies the condition $h^0(\mathcal{G}(-1)) \leq 1$.*

Proof. Let $\mathcal{G}$ be in $\mathrm{M}_{\mathbb{P}^2}(5,2)$ and assume that $h^0(\mathcal{G}(-1)) > 0$. As in the proof of 2.1.3 [4], there is an injective morphism $\mathcal{O}_C \to \mathcal{G}(-1)$ for a curve $C \subset \mathbb{P}^2$. From the semi-stability of $\mathcal{G}(-1)$ we see that C must be a quintic curve. The quotient sheaf $\mathcal{G}(-1)/\mathcal{O}_C$ is a sheaf of dimension zero and length 2; it maps surjectively onto the structure sheaf $\mathbb{C}_x$ of a point x. Let $\mathcal{G}'$ be the kernel of the composed morphism $\mathcal{G} \to \mathbb{C}_x$. If $\mathcal{G}'$ is semi-stable, then, from 3.1.5, we have $h^0(\mathcal{G}'(-1)) \leq 1$. It follows that $h^0(\mathcal{G}(-1)) \leq 1$ unless $h^0(\mathcal{G}'(-1)) = 1$ and the morphism $\mathcal{G}(-1) \to \mathbb{C}_x$ is surjective on global sections. In this case we can apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{G}'(-1) \longrightarrow \mathcal{G}(-1) \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

to the standard resolution of $\mathbb{C}_x$ and to resolution 3.1.5 for $\mathcal{G}'$ tensored with $\mathcal{O}(-1)$, which reads:

$$0 \longrightarrow 2\mathcal{O}(-4) \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O} \longrightarrow \mathcal{G}'(-1) \longrightarrow 0.$$

We get a resolution of the form

$$0 \longrightarrow 2\mathcal{O}(-4) \oplus \mathcal{I}_x \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O} \longrightarrow \mathcal{G}(-1) \longrightarrow 0.$$

We now arrive at a contradiction as in the proof of 2.1.4. The image of $\mathcal{I}_x$ is included in a factor $\mathcal{O}$ of $2\mathcal{O}$. As $\mathcal{G}(-1)$ has no zero-dimensional torsion, this factor $\mathcal{O}$ maps to zero in $\mathcal{G}(-1)$, which is absurd.

Assume now that $\mathcal{G}'$ is not semi-stable and let $\mathcal{G}'' \subset \mathcal{G}'$ be a destabilising subsheaf. We may assume that $\mathcal{G}''$ itself is semi-stable, say it gives a point in $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$. We have the inequalities

$$\frac{1}{5} = p(\mathcal{G}') < \frac{\chi}{r} < p(\mathcal{G}) = \frac{2}{5}$$

leaving only the possibilities $(r,\chi) = (4,1)$ or $(3,1)$. Denote $\mathcal{C} = \mathcal{G}/\mathcal{G}''$. If $\mathcal{G}''$ is in $\mathrm{M}_{\mathbb{P}^2}(4,1)$, then $P_{\mathcal{C}}(t) = t+1$. Moreover, the zero-dimensional torsion of $\mathcal{C}$ vanishes, otherwise its pull-back in $\mathcal{G}$ would be a destabilising subsheaf. We deduce that $\mathcal{C} = \mathcal{O}_L$ for a line $L \subset \mathbb{P}^2$. But $h^0(\mathcal{O}_L(-1)) = 0$ and, according to 2.1.3 [4], also $h^0(\mathcal{G}''(-1)) = 0$. We get $h^0(\mathcal{G}(-1)) = 0$, contradicting our hypothesis on $\mathcal{G}$.

The last case to examine is when $\mathcal{G}''$ is in $\mathrm{M}_{\mathbb{P}^2}(3,1)$. We have $P_{\mathcal{C}}(t) = 2t+1$. As before, $\mathcal{C}$ has no zero-dimensional torsion. Moreover, any quotient sheaf destabilising $\mathcal{C}$ must also destabilise $\mathcal{G}$. We conclude that $\mathcal{C}$ is semi-stable, i.e. $\mathcal{C} = \mathcal{O}_C$ for a conic curve $C \subset \mathbb{P}^2$. But $h^0(\mathcal{O}_C(-1)) = 0$ and, according to 2.1.3 [4], also $h^0(\mathcal{G}''(-1)) = 0$. We conclude that $h^0(\mathcal{G}(-1)) = 0$, contrary to our hypothesis on $\mathcal{G}$. □

Proposition 2.1.6. *There are no sheaves $\mathcal{G}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(5,2)$ and satisfying the conditions $h^0(\mathcal{G}(-1)) = 1$ and $h^1(\mathcal{G}) \geq 2$.*

Proof. Fix an integer $m \geq 0$ and let X be the set of sheaves $\mathcal{G}$ in $\mathrm{M}_{\mathbb{P}^2}(5,2)$ satisfying $h^0(\mathcal{G}(-1)) = 1$ and $h^0(\mathcal{G} \otimes \Omega^1) = m$. Let $Y \subset X$ be the subset of sheaves satisfying the additional condition $h^1(\mathcal{G}) = 1$. According to 2.1.3 [4], for every sheaf in X we have $\mathrm{H}^0(\mathcal{G}(-2)) = 0$. The Beilinson free monad (2.2.1) [4] for $\mathcal{G}(-1)$ reads

$$0 \longrightarrow 8\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow (m+11)\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow 4\mathcal{O} \longrightarrow 0.$$

Thus X is parametrised by an open subset M inside the space of monads of the form

$$0 \longrightarrow 8\mathcal{O}(-1) \oplus m\mathcal{O} \xrightarrow{A} (m+11)\mathcal{O} \oplus \mathcal{O}(1) \xrightarrow{B} 4\mathcal{O}(1) \longrightarrow 0,$$

where $A_{12} = 0$, $B_{12} = 0$. Let Γ be the space of pairs (A,B) of morphisms

$$A \in \mathrm{Hom}(8\mathcal{O}(-1) \oplus m\mathcal{O}, (m+11)\mathcal{O} \oplus \mathcal{O}(1)),$$
$$B \in \mathrm{Hom}((m+11)\mathcal{O} \oplus \mathcal{O}(1), 4\mathcal{O}(1)),$$

such that A is injective, B is surjective, $A_{12} = 0$, $B_{12} = 0$. Consider the algebraic map $\gamma\colon \Gamma \to \mathrm{Hom}(8\mathcal{O}(-1), 4\mathcal{O}(1))$ given by $\gamma(A,B) = B_{11} \circ A_{11}$. Note that M is an open subset inside $\gamma^{-1}(0)$. We claim that M is smooth. For this it is sufficient to show that γ has surjective differential at every point of M. The tangent space of Γ at an arbitrary point (A,B) is the space of

pairs (α,β) of morphisms

$$\alpha \in \mathrm{Hom}(8\mathcal{O}(-1)\oplus m\mathcal{O}, (m+11)\mathcal{O}\oplus\mathcal{O}(1)),$$
$$\beta \in \mathrm{Hom}((m+11)\mathcal{O}\oplus\mathcal{O}(1), 4\mathcal{O}(1)),$$

such that $\alpha_{12}=0$, $\beta_{12}=0$. We have $\mathrm{d}\gamma_{(A,B)}(\alpha,\beta) = B_{11}\circ\alpha_{11}+\beta_{11}\circ A_{11}$. It is enough to prove that the map $\alpha_{11}\to B_{11}\circ\alpha_{11}$ is surjective at a point $(A,B)\in M$. For this we apply the long $\mathrm{Ext}(8\mathcal{O}(-1), \_\_)$-sequence to the exact sequence

$$0 \longrightarrow \mathcal{K}er(B_{11}) \longrightarrow (m+11)\mathcal{O} \xrightarrow{B_{11}} 4\mathcal{O}(1) \longrightarrow 0$$

and we use the vanishing of $\mathrm{Ext}^1(8\mathcal{O}(-1), \mathcal{K}er(B_{11}))$. This vanishing follows from the exact sequence

$$0 \longrightarrow 8\mathcal{O}(-1)\oplus m\mathcal{O} \longrightarrow \mathcal{K}er(B_{11})\oplus\mathcal{O}(1) \longrightarrow \mathcal{G} \longrightarrow 0$$

and the vanishing of $\mathrm{H}^1(\mathcal{G}(1))$, which is a consequence of 2.1.3 [4].

Let $\upsilon\colon M\to X$ be the surjective morphism which sends a monad to the isomorphism class of its cohomology. The tangent space to M at an arbitrary point (A,B) is

$$\mathbb{T}_{(A,B)}M = \{(\alpha,\beta) \mid \alpha_{12}=0, \beta_{12}=0, \beta\circ A + B\circ\alpha = 0\}.$$

Consider the map $\Phi\colon M\to \mathrm{Hom}((m+11)\mathcal{O}, 4\mathcal{O}(1))$, $\Phi(A,B)=B_{11}$. It has surjective differential at every point. Indeed, $\mathrm{d}\Phi_{(A,B)}(\alpha,\beta)=\beta_{11}$, so we need to show that, given β_{11}, there is α such that $\beta\circ A+B\circ\alpha=0$, that is $-\beta_{11}\circ A_{11} = B_{11}\circ\alpha_{11}$. This follows from the surjectivity of the map $\alpha_{11}\to B_{11}\circ\alpha_{11}$, which we proved above.

We have $h^0(\mathcal{G}) = 14 - \mathrm{rank}(\mathrm{H}^0(B_{11}))$. The subset $N\subset M$ of monads with cohomology $\mathcal{G}$ satisfying $h^1(\mathcal{G})\geq 2$ is the preimage under Φ of the set of morphisms of rank at most 10. Since any matrix of rank at most 10 is the limit of a sequence of matrices of rank 11, and since the derivative of Φ is surjective at every point, we deduce that N is included in $\overline{\upsilon^{-1}(Y)}\setminus\upsilon^{-1}(Y)$. But, according to 2.1.4, Y is empty for $m\neq 0$. For $m=0$, we shall prove at 2.2.6 below that Y is closed. We conclude that N is empty. □

2.2. **Description of the strata as quotients.** In subsection 2.1 we found that the moduli space $\mathrm{M}_{\mathbb{P}^2}(5,3)$ can be decomposed into four strata:

- an open stratum X_0 given by the conditions $h^0(\mathcal{F}(-1))=0$, $h^0(\mathcal{F}\otimes\Omega^1(1))=1$;
- a locally closed stratum X_1 of codimension 2 given by the conditions $h^0(\mathcal{F}(-1))=0$, $h^0(\mathcal{F}\otimes\Omega^1(1))=2$;
- a locally closed stratum X_2 of codimension 3 given by the conditions $h^0(\mathcal{F}(-1))=1$, $h^1(\mathcal{F})=0$;
- the stratum X_3 of codimension 4 given by the conditions $h^0(\mathcal{F}(-1))=1$, $h^1(\mathcal{F})=1$. We shall see below at 2.2.6 that X_3 is closed.

In the sequel X_i will be equipped with the canonical induced reduced structure. Let W_0, W_1, W_2, W_3 be the sets of morphisms φ from 2.1.2(i), 2.1.2(ii), 2.1.3, respectively 2.1.4. Each sheaf $\mathcal{F}$ giving a point in X_i is the cokernel of a morphism $\varphi \in W_i$. Let $\mathbb{W}_i = \operatorname{Hom}(\mathcal{A}_i, \mathcal{B}_i)$ denote the ambient vector space containing W_i. Here $\mathcal{A}_i$, $\mathcal{B}_i$ are locally free sheaves on $\mathbb{P}^2$, for instance $\mathcal{A}_0 = 2\mathcal{O}(-2) \oplus \mathcal{O}(-1)$, $\mathcal{B}_0 = 3\mathcal{O}$. The natural group of automorphisms $G_i = (\operatorname{Aut}(\mathcal{A}_i) \times \operatorname{Aut}(\mathcal{B}_i))/\mathbb{C}^*$ acts on $\mathbb{W}_i$ by conjugation, leaving W_i invariant (here $\mathbb{C}^*$ is embedded as the subgroup of homotheties). In this subsection we shall prove that there exist geometric quotients W_i/G_i, which are smooth quasiprojective varieties (W_3/G_3 is even projective), such that $W_i/G_i \simeq X_i$. Whenever possible, we shall give concrete descriptions of these quotients.

Proposition 2.2.1. *There exists a geometric quotient W_0/G_0, which is a smooth quasiprojective variety. W_0/G_0 is isomorphic to X_0.*

Proof. Let $\Lambda = (\lambda_1, \lambda_2, \mu_1)$ be a polarisation for the action of G_0 on $\mathbb{W}_0$ satisfying $1/6 < \lambda_1 < 1/3$ (see [5] for the notions of polarisation and of semi-stable morphism). According to 4.3 [9], W_0 is the open invariant subset of injective morphisms inside the set $\mathbb{W}_0^{ss}(\Lambda)$ of semi-stable morphisms with respect to Λ. According to 6.4 [3], if $\lambda_1 < 1/5$, then there is a geometric quotient $\mathbb{W}_0^{ss}(\Lambda)/G_0$, which is a projective variety (see also 7.11 [9]). We fix Λ satisfying $1/6 < \lambda_1 < 1/5$. It is now clear that a geometric quotient W_0/G_0 exists and is an open subset of $\mathbb{W}_0^{ss}(\Lambda)/G_0$.

The morphism $W_0 \to X_0$ sending φ to the isomorphism class of $\mathcal{C}oker(\varphi)$ is surjective and its fibres are G_0-orbits, hence it factors through a bijective morphism $W_0/G_0 \to X_0$. Since X_0 is smooth, Zariski's Main Theorem tells us that the latter is an isomorphism. $\square$

We remark that W_0 is a proper subset of $\mathbb{W}_0^{ss}(\Lambda)$, hence W_0/G_0 is a proper open subset of the projective variety $\mathbb{W}_0^{ss}(\Lambda)/G_0$. Indeed, the morphism φ_0 represented by the matrix

$$\left[\begin{array}{ccc} XY & X^2 & 0 \\ XZ & 0 & X \\ 0 & -XZ & Y \end{array} \right]$$

is not injective but is semi-stable with respect to Λ. This follows from King's criterion of semi-stability [7], which, in our case, says that a morphism is in $\mathbb{W}_0^{ss}(\Lambda)$ if and only if it is not equivalent to a morphism having one of the following forms:

$$\left[\begin{array}{ccc} \star & \star & 0 \\ \star & \star & 0 \\ \star & \star & \star \end{array} \right], \quad \left[\begin{array}{ccc} \star & 0 & 0 \\ \star & \star & \star \\ \star & \star & \star \end{array} \right], \quad \left[\begin{array}{ccc} 0 & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{array} \right], \quad \left[\begin{array}{ccc} 0 & 0 & \star \\ 0 & 0 & \star \\ \star & \star & \star \end{array} \right].$$

The first case is excluded by the fact that φ_0 has two linearly independent entries on column 3, the second case is excluded by the fact that φ_0 has two

linearly independent entries on row 1. To exclude the third case assume that

$$\left[\begin{array}{ccc} XY & X^2 & 0 \\ XZ & 0 & X \\ 0 & -XZ & Y \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \\ \ell \end{array}\right]=\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right],$$

for $c_1, c_2 \in \mathbb{C}$ and $\ell \in V^*$. Then the triple (c_1X, c_2X, ℓ) is a multiple of $(-X, Y, Z)$, which is absurd. The last case can also be easily excluded.

We recall from 2.4 [4] the moduli spaces $\mathrm{N}(3, m, n)$ of semi-stable Kronecker modules $f\colon \mathbb{C}^m \otimes V \to \mathbb{C}^n$.

Proposition 2.2.2. *There exists a geometric quotient W_1/G_1 and it is a proper open subset inside a fibre bundle over $\mathbb{P}^2 \times \mathrm{N}(3,2,3)$ with fibre $\mathbb{P}^{16}$.*

Proof. Let W_1' be the locally closed subset of $\mathbb{W}_1$ given by the conditions that $\varphi_{12} = 0$, φ_{11} have linearly independent entries and φ_{22} have linearly independent maximal minors. The set of morphisms φ_{11} form an open subset $U_1 \subset \mathrm{Hom}(2\mathcal{O}(-2), \mathcal{O}(-1))$ and the set of morphisms φ_{22} form an open subset $U_2 \subset \mathrm{Hom}(2\mathcal{O}(-1), 3\mathcal{O})$. We denote $U = U_1 \times U_2$. W_1' is the trivial vector bundle over U with fibre $\mathrm{Hom}(2\mathcal{O}(-2), 3\mathcal{O})$. We represent the elements of G_1 by pairs of matrices

$$(g,h) \in \mathrm{Aut}(2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1)) \times \mathrm{Aut}(\mathcal{O}(-1) \oplus 3\mathcal{O}),$$

$$g = \left[\begin{array}{cc} g_1 & 0 \\ u & g_2 \end{array}\right], \qquad h = \left[\begin{array}{cc} h_1 & 0 \\ v & h_2 \end{array}\right].$$

Inside G_1 we distinguish three subgroups: a unitary subgroup G_1' given by the conditions that g_1, g_2, h_1, h_2 be the identity morphisms, a reductive subgroup $G_{1\mathrm{red}}$ given by the conditions $u = 0$, $v = 0$ and a subgroup S of $G_{1\mathrm{red}}$ isomorphic to $\mathbb{C}^*$ given by the conditions that g_1, h_1 be the morphisms of multiplication by a non-zero constant a and that g_2, h_2 be the morphisms of multiplication by a non-zero constant b. Note that $G_1 = G_1' G_{1\mathrm{red}}$. Consider the G_1-invariant subset $\Sigma \subset W_1'$ given by the condition

$$\varphi_{21} = \varphi_{22}u + v\varphi_{11}, \quad u \in \mathrm{Hom}(2\mathcal{O}(-2), 2\mathcal{O}(-1)), \quad v \in \mathrm{Hom}(\mathcal{O}(-1), 3\mathcal{O}).$$

Note that W_1 is the subset of injective morphisms inside $W_1' \setminus \Sigma$, so it is open and G_1-invariant. Moreover, it is a proper subset as, for instance, the morphism represented by the matrix

$$\left[\begin{array}{cccc} Y & X & 0 & 0 \\ 0 & Y^2 & X & 0 \\ 0 & YZ & 0 & X \\ 0 & 0 & -Z & Y \end{array}\right]$$

is in $W_1' \setminus \Sigma$ but is not injective. Our aim is to construct a geometric quotient of $W_1' \setminus \Sigma$ modulo G_1; it will follow that W_1/G_1 exists and is a proper open subset of $(W_1' \setminus \Sigma)/G_1$.

Firstly, we construct the geometric quotient W_1'/G_1'. Because of the conditions on φ_{11} and φ_{22} it is easy to check that Σ is a subbundle of W_1'. The quotient bundle, denoted Q', has rank 17. The quotient map $W_1' \to Q'$ is a geometric quotient modulo G_1'. Moreover, the canonical action of $G_{1\mathrm{red}}$ on U is Q'-linearised and the map $W_1' \to Q'$ is $G_{1\mathrm{red}}$-equivariant. Let σ be the zero-section of Q'. The restricted map $W_1' \setminus \Sigma \to Q' \setminus \sigma$ is also a geometric quotient map modulo G_1'.

Let $x \in U$ be a point and let $\xi \in Q_x'$ be a non-zero vector lying over x. The stabiliser of x in $G_{1\mathrm{red}}$ is S and $S\xi = \mathbb{C}^*\xi$. Thus the canonical map $Q' \setminus \sigma \to \mathbb{P}(Q')$ is a geometric quotient modulo S. It remains to construct a geometric quotient of $\mathbb{P}(Q')$ modulo the induced action of $G_{1\mathrm{red}}/S$.

The existence of a geometric quotient of U modulo $G_{1\mathrm{red}}/S$ follows from the classical geometric invariant theory. We notice that

$$G_{1\mathrm{red}}/S \simeq ((\mathrm{Aut}(2\mathcal{O}(-2)) \times \mathrm{Aut}(\mathcal{O}(-1)))/\mathbb{C}^*)\times \\ ((\mathrm{Aut}(2\mathcal{O}(-1)) \times \mathrm{Aut}(3\mathcal{O}))/\mathbb{C}^*).$$

Using King's criterion of semi-stability [7], we can see that U_1 is the set of semi-stable points for the canonical action by conjugation of

$$(\mathrm{Aut}(2\mathcal{O}(-2)) \times \mathrm{Aut}(\mathcal{O}(-1)))/\mathbb{C}^* \quad \text{on} \quad \mathrm{Hom}(2\mathcal{O}(-2), \mathcal{O}(-1)).$$

The resulting geometric quotient is $\mathrm{N}(3, 2, 1)$ and is clearly isomorphic to $\mathbb{P}^2$. Analogously, U_2 is the set of semi-stable points for the action of

$$(\mathrm{Aut}(2\mathcal{O}(-1)) \times \mathrm{Aut}(3\mathcal{O}))/\mathbb{C}^* \quad \text{on} \quad \mathrm{Hom}(2\mathcal{O}(-1), 3\mathcal{O})$$

and the resulting quotient is $\mathrm{N}(3, 2, 3)$. According to [1], this is a smooth projective irreducible variety of dimension 6. We obtain:

$$U/(G_{1\mathrm{red}}/S) \simeq \mathrm{N}(3, 2, 1) \times \mathrm{N}(3, 2, 3) \simeq \mathbb{P}^2 \times \mathrm{N}(3, 2, 3).$$

It remains to show that $\mathbb{P}(Q')$ descends to a fibre bundle over $U/(G_{1\mathrm{red}}/S)$. We consider the character χ of $G_{1\mathrm{red}}$ given by $\chi(g, h) = \det(g)\det(h)^{-1}$. Note that χ is well-defined because it is trivial on homotheties. We multiply the action of $G_{1\mathrm{red}}$ on Q' by χ and we denote the resulting linearised bundle by Q_χ'. The action of S on Q_χ' is trivial, hence Q_χ' is $G_{1\mathrm{red}}/S$-linearised. The isotropy subgroup in $G_{1\mathrm{red}}/S$ for any point in U is trivial, so we can apply [6], lemma 4.2.15, to deduce that Q_χ' descends to a vector bundle Q over $U/(G_{1\mathrm{red}}/S)$. The induced map $\mathbb{P}(Q') \to \mathbb{P}(Q)$ is a geometric quotient map modulo $G_{1\mathrm{red}}/S$. We conclude that the composed map

$$W_1' \setminus \Sigma \longrightarrow Q' \setminus \sigma \longrightarrow \mathbb{P}(Q') \longrightarrow \mathbb{P}(Q)$$

is a geometric quotient map modulo G_1 and that a geometric quotient W_1/G_1 exists and is a proper open subset inside $\mathbb{P}(Q)$. □

Proposition 2.2.3. *The geometric quotient W_1/G_1 is isomorphic to X_1.*

Proof. As at 2.2.1, we have a canonical bijective morphism $W_1/G_1 \to X_1$. To show that this is an isomorphism we shall use the method of 3.1.6 [4]. Our aim is to construct resolution 2.1.2(ii) not merely for an individual sheaf giving a point in X_1, but also for a flat family of sheaves giving points in X_1. We achieve this for local flat families by obtaining resolution 2.1.2(ii) in a natural manner from the relative Beilinson spectral sequence associated to the family. Thus, for any sheaf $\mathcal{F}$ giving a point in X_1, we need to recover its resolution from its Beilinson spectral sequence. Diagram (2.2.3) [4] for $\mathcal{F}$ reads:

$$\begin{array}{ccccc} 2\mathcal{O}(-2) & \xrightarrow{\varphi_1} & \mathcal{O}(-1) & & 0 \quad . \\ & & & & \\ 0 & & 2\mathcal{O}(-1) & \xrightarrow{\varphi_4} & 3\mathcal{O} \end{array}$$

Since $\mathcal{F}$ is semi-stable and maps surjectively onto $\mathcal{C}oker(\varphi_1)$, we see that $\mathcal{C}oker(\varphi_1)$ is the structure sheaf $\mathbb{C}_x$ of a point $x \in \mathbb{P}^2$ and that $\mathcal{K}er(\varphi_1)$ is isomorphic to $\mathcal{O}(-3)$. The exact sequence (2.2.5) [4]

$$0 \longrightarrow \mathcal{K}er(\varphi_1) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}oker(\varphi_1) \longrightarrow 0$$

gives the extension

$$0 \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}oker(\varphi_1) \longrightarrow 0.$$

We apply the horseshoe lemma to the above extension and to the resolutions

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \longrightarrow 3\mathcal{O} \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{C}oker(\varphi_1) \longrightarrow 0.$$

We arrive at the exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Since $\mathrm{H}^1(\mathcal{F}) = 0$, we see that $\mathcal{O}(-3)$ can be cancelled and we get resolution 2.1.2(ii), as desired. □

Proposition 2.2.4. *There exists a geometric quotient W_2/G_2, which is a proper open subset inside a fibre bundle over* $\mathrm{N}(3,3,2)$ *with fibre* $\mathbb{P}^{17}$*. Moreover, W_2/G_2 is isomophic to X_2.*

Proof. The existence of W_2/G_2 follows from the construction of quotients given at 9.3 [5]. Our situation is also analogous to 3.1.2 [4]. We consider a polarisation $\Lambda = (\lambda_1, \mu_1, \mu_2)$ as in [5] for the action of G_2 on $\mathbb{W}_2$ satisfying the condition $0 < \mu_2 < 1/3$. According to op.cit., lemma 9.3.1, the open subset $\mathbb{W}_2^{ss}(\Lambda) \subset \mathbb{W}_2$ of semi-stable morphisms with respect to Λ is the set of morphisms φ for which of φ_{11} is semi-stable with respect to the action by conjugation of $(\mathrm{GL}(3,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}))/\mathbb{C}^*$ on $\mathrm{Hom}(3\mathcal{O}(-2), 2\mathcal{O}(-1))$ and such that φ is not equivalent to a morphism ψ satisfying $\psi_{21} = 0$. According to King's criterion of semi-stability [7], the condition on φ_{11} is the same as saying

that φ_{11} is not equivalent to a morphism represented by a matrix having a zero-column or a zero-submatrix of size 1×2. Furthermore, this is equivalent to the condition on φ_{11} from 2.1.3. We see now that W_2 is the open invariant subset of injective morphisms inside $\mathbb{W}_2^{ss}(\Lambda)$. It is a proper subset because it is easy to construct semi-stable morphisms that are not injective, for example the morphism represented by the matrix

$$\begin{bmatrix} 0 & X & Y \\ X & 0 & -Z \\ Y^3 & ZY^2 & 0 \end{bmatrix}.$$

Adopting the notations of 3.1.2 [4], let $\mathrm{N}(3,3,2)$ be the moduli space of semi-stable Kronecker modules $f\colon 3\mathcal{O}(-2) \to 2\mathcal{O}(-1)$, let $\tau\colon E \otimes V \to F$ be the universal morphism on $\mathrm{N}(3,3,2)$, let p_1, p_2 be the projections of $\mathrm{N}(3,3,2) \times \mathbb{P}^2$ onto its factors and let

$$\theta\colon p_1^*(E) \otimes p_2^*(\mathcal{O}(-2)) \longrightarrow p_1^*(F) \otimes p_2^*(\mathcal{O}(-1))$$

be the morphism induced by τ. The sheaf $\mathcal{U} = p_{1*}(\mathcal{C}oker(\theta^*) \otimes p_2^*\mathcal{O}(1))$ is locally free on $\mathrm{N}(3,3,2)$ of rank 18. According to 9.3 [5], $\mathbb{P}(\mathcal{U})$ is a geometric quotient of $\mathbb{W}_2^{ss}(\Lambda)$ modulo G_2. Thus W_2/G_2 exists and is a proper open subset of $\mathbb{P}(\mathcal{U})$.

We shall now prove that the natural bijective morphism $W_2/G_2 \to X_2$ is an isomorphism. Given $\mathcal{F}$ in X_2, we need to construct resolution 2.1.3 starting from the Beilinson spectral sequence of $\mathcal{F}$. It is easier to work, instead, with the dual sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$, which gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,2)$. The Beilinson tableau (2.2.3) [4] for $\mathcal{G}$ takes the form

$$\begin{array}{ccc} 3\mathcal{O}(-2) \xrightarrow{\varphi_1} & 3\mathcal{O}(-1) \xrightarrow{\varphi_2} & \mathcal{O} \\ & & \\ 0 & 2\mathcal{O}(-1) \xrightarrow{\varphi_4} & 3\mathcal{O} \end{array}.$$

According to 2.2 [4], φ_2 is surjective while φ_4 is injective. Thus $\mathcal{K}er(\varphi_2) \simeq \Omega^1$. Consider the canonical morphism

$$\rho\colon 3\mathcal{O}(-2) \simeq \mathcal{O}(-2) \otimes \mathrm{Hom}(\mathcal{O}(-2), \Omega^1) \longrightarrow \Omega^1.$$

There is a morphism $\alpha\colon 3\mathcal{O}(-2) \to 3\mathcal{O}(-2)$ such that $\rho \circ \alpha = \varphi_1$. Since $\mathcal{G}$ maps surjectively onto $\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1)$, this sheaf has rank zero, i.e. $\mathcal{I}m(\varphi_1)$ has rank 2. This excludes the possibility $\mathrm{rank}(\alpha) = 1$, because in this case $\mathcal{I}m(\varphi_1)$ would be isomorphic to $\mathcal{O}(-2)$. If $\mathrm{rank}(\alpha) = 2$, then $\mathcal{I}m(\varphi_1)$ would be isomorphic to $2\mathcal{O}(-2)$. In this case $\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1)$ would have slope -1, hence it would destabilise $\mathcal{G}$. We deduce that $\mathrm{rank}(\alpha) = 3$, hence $\mathcal{I}m(\varphi_1) = \mathcal{K}er(\varphi_2)$ and $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$. The exact sequence (2.2.5) [4] takes the form

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{G} \longrightarrow 0.$$

This easily yields the dual to resolution 2.1.3. □

Proposition 2.2.5. *There exists a geometric quotient W_3/G_3 and it is a smooth projective variety. W_3/G_3 is isomorphic to the Hilbert flag scheme of quintic curves in $\mathbb{P}^2$ containing zero-dimensional subschemes of length 2.*

Proof. Before constructing the quotient we notice that its existence already follows from [5]. Let $\Lambda = (\lambda_1, \lambda_2, \mu_1, \mu_2)$ be a polarisation for the action of G_3 on $\mathbb{W}_3$, as in loc.cit. Using King's criterion of semi-stability [7] we can verify that for polarisations satisfying $\lambda_1 < \mu_1$ and $\lambda_1 < \mu_2$ the set of stable points $\mathbb{W}_3^s(\Lambda)$ coincides with the set of semi-stable points $\mathbb{W}_3^{ss}(\Lambda)$ and is equal to W_3. According to [5], for polarisations satisfying $\lambda_2 > 6\lambda_1$ and $\mu_1 > 3\mu_2$ there is a good and projective quotient $\mathbb{W}_3^{ss}(\Lambda)//G_3$ containing the smooth geometric quotient $\mathbb{W}_3^s(\Lambda)/G_3$ as an open subset. We now choose a polarisation satisfying all the above conditions, i.e. satisfying $0 < \lambda_1 < 1/7$ and $\lambda_1 < \mu_2 < 1/4$. We conclude that there is a smooth geometric quotient W_3/G_3, which is a projective variety.

Next we give two constructions of W_3/G_3, firstly as a bundle and secondly as a Hilbert flag scheme. The first construction uses the method of 2.2.2, which consisted of finding successively quotients modulo subgroups. Let W_3' be the open subset of $\mathbb{W}_3$ given by the conditions that $\varphi_{12} \neq 0$ and that φ_{22} be non-divisible by φ_{12}. The pairs of morphisms $(\varphi_{12}, \varphi_{22})$ form an open subset $U \subset \operatorname{Hom}(\mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}(1))$ and W_3' is the trivial vector bundle over U with fibre $\operatorname{Hom}(\mathcal{O}(-3), \mathcal{O} \oplus \mathcal{O}(1))$. We represent the elements of G_3 by pairs of matrices

$$(g, h) \in \operatorname{Aut}(\mathcal{O}(-3) \oplus \mathcal{O}(-1)) \times \operatorname{Aut}(\mathcal{O} \oplus \mathcal{O}(1)),$$
$$g = \begin{bmatrix} g_1 & 0 \\ u & g_2 \end{bmatrix}, \qquad h = \begin{bmatrix} h_1 & 0 \\ v & h_2 \end{bmatrix}.$$

Inside G_3 we distinguish two subgroups: a unitary subgroup G_3' given by the conditions that h be the identity morphism, $g_1 = 1$, $g_2 = 1$ and a subgroup G_3'' given by the condition that g be the identity morphism. Consider the G_3-invariant subset $\Sigma \subset W_3'$ given by the conditions

$$\varphi_{11} = \varphi_{12}u, \quad \varphi_{21} = \varphi_{22}u, \quad u \in \operatorname{Hom}(\mathcal{O}(-3), \mathcal{O}(-1)).$$

Note that $W_3 = W_3' \setminus \Sigma$. Clearly Σ is a subbundle of W_3'. The quotient bundle E' has rank 19. The quotient map $W_3' \to E'$ is a geometric quotient modulo G_3'. Moreover, the canonical action of G_3'' on U is E'-linearised and the map $W_3' \to E'$ is G_3''-equivariant. Let σ' be the zero-section of E'. The restricted map $W_3 \to E' \setminus \sigma'$ is also a geometric quotient modulo G_3'.

We now construct a geometric quotient of E' modulo G_3''. The quotient for the base U can be described explicitly as follows. On $\mathbb{P}(V^*)$ we consider the trivial vector bundle with fibre S^2V^* and the subbundle with fibre vV^* at any point $\langle v \rangle \in \mathbb{P}(V^*)$. Let Q be the quotient bundle. Clearly U/G_3'' is isomorphic to $\mathbb{P}(Q)$. Moreover, U is a principal G_3''-bundle over $\mathbb{P}(Q)$.

According to 4.2.14 [6], E' descends to a vector bundle E on $\mathbb{P}(Q)$. E is the geometric quotient E'/G_3''. Let σ be the zero-section of E. The composed map $W_3 \to E' \setminus \sigma' \to E \setminus \sigma$ is a geometric quotient modulo $G_3'G_3''$. It is now clear that the fibre bundle $\mathbb{P}(E)$ is the geometric quotient W_3/G_3. Thus W_3/G_3 is a fibre bundle with fibre $\mathbb{P}^{18}$ and base a fibre bundle $\mathbb{P}(Q)$ with base $\mathbb{P}^2$ and fibre $\mathbb{P}^2$.

It is clear that $\mathbb{P}(Q)$ is isomorphic to the Hilbert scheme of zero-dimensional subschemes of $\mathbb{P}^2$ of length 2. Let F be the Hilbert flag scheme from the proposition viewed as a subscheme of $\mathbb{P}(Q) \times \mathbb{P}(S^5V^*)$. Consider the map $W_3 \to F$ defined by

$$\varphi \longrightarrow (\langle \varphi_{12} \rangle, \langle \varphi_{22} \mod \varphi_{12} \rangle, \langle \det(\varphi) \rangle).$$

The fibres of this map are obviously G_3-orbits. To show that this map is a geometric quotient we shall construct local sections. We choose a point $x = (\langle f \rangle, \langle g \mod f \rangle, \langle h \rangle)$ in F. To fix notations we write $f = X$ and we may assume that g is a quadratic form in Y and Z. There are unique forms $h_1(Y, Z)$ and $h_2(X, Y, Z)$ such that $h = h_1 + Xh_2$. By hypothesis h_1 is divisible by g. We put

$$\sigma(x) = \begin{bmatrix} h_1/g & f \\ -h_2 & g \end{bmatrix}.$$

Note that σ extends to a local section in a neighbourhood of x because h_2 and h_1, hence also h_1/g, depend algebraically on x. □

Proposition 2.2.6. *The geometric quotient W_3/G_3 is isomorphic to X_3. In particular, X_3 is a smooth closed subvariety of $\mathrm{M}_{\mathbb{P}^2}(5, 3)$.*

Proof. As above, in order to show that the bijective morphism $W_3/G_3 \to X_3$ is an isomorphism, we need to construct resolution 2.1.4 starting from the Beilinson tableau (2.2.3) [4] for $\mathcal{F}$, which takes the form:

$$3\mathcal{O}(-2) \xrightarrow{\varphi_1} 3\mathcal{O}(-1) \xrightarrow{\varphi_2} \mathcal{O} \;.$$

$$\mathcal{O}(-2) \xrightarrow{\varphi_3} 4\mathcal{O}(-1) \xrightarrow{\varphi_4} 4\mathcal{O}$$

As at 2.2.4, $\mathcal{K}er(\varphi_2)$ is equal to $\mathcal{I}m(\varphi_1)$ and $\mathcal{K}er(\varphi_1)$ is isomorphic to $\mathcal{O}(-3)$. The exact sequence (2.2.5) [4] gives the resolution

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{F} \longrightarrow 0.$$

We combine this sequence with the exact sequence (2.2.4) [4] that reads as follows:

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\varphi_3} 4\mathcal{O}(-1) \xrightarrow{\varphi_4} 4\mathcal{O} \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow 0.$$

Indeed, φ_5 lifts to a map $\mathcal{O}(-3) \to 4\mathcal{O}$ because $\mathrm{Ext}^1(\mathcal{O}(-3), \mathcal{C}oker(\varphi_3)) = 0$. We arrive at the resolution

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-3) \oplus 4\mathcal{O}(-1) \longrightarrow 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

We have already seen at 2.1.4 how to derive the desired resolution of $\mathcal{F}$ from the above exact sequence. $\square$

2.3. **Geometric description of the strata.** We recall that the stratum X_0 of $\mathrm{M}_{\mathbb{P}^2}(5,3)$ consists of isomorphism classes of cokernels of morphisms $\varphi = (\varphi_{11}, \varphi_{12})$ as at 2.1.2(i). We distinguish a subset $X_{01} \subset X_0$ given by the condition $\mathcal{C}oker(\varphi_{12}) \simeq \mathcal{I}_x(1) \oplus \mathcal{O}$, where $\mathcal{I}_x \subset \mathcal{O}$ is the ideal sheaf of a point $x \in \mathbb{P}^2$. Clearly X_{01} is closed in X_0 and has codimension 1.

Proposition 2.3.1. *The sheaves $\mathcal{F}$ giving points in $X_0 \setminus X_{01}$ are precisely the sheaves admitting a resolution of the form*

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow \Omega^1(2) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Proof. Assume that $\mathcal{F}$ gives a point in $X_0 \setminus X_{01}$. From 2.1.2(i) we have the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{C}oker(\varphi_{12}) \longrightarrow \mathcal{F} \longrightarrow 0.$$

By hypothesis $\mathcal{C}oker(\varphi_{12})$ is isomorphic to $\Omega^1(2)$.

Assume now that $\mathcal{F}$ has a resolution as in the proposition. Combining with the Euler sequence we find an injective morphism $\varphi\colon 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \to 3\mathcal{O}$ such that $\mathcal{F} \simeq \mathcal{C}oker(\varphi)$. The fact that φ_{12} has linearly independent entries ensures that φ satisfies the conditions from 2.1.2(i). $\square$

Proposition 2.3.2. *The generic sheaves $\mathcal{F}$ from X_{01} are precisely the non-split extension sheaves of the form*

$$0 \longrightarrow \mathcal{J}_x(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

such that there is a global section of $\mathcal{F}$ taking the value 1 at every point of Z. Here $\mathcal{J}_x \subset \mathcal{O}_C$ is the ideal sheaf of a point x on a quintic curve $C \subset \mathbb{P}^2$ and $Z \subset C$ is a union of four distinct points, also distinct from x, no three of which are colinear.

There is an open subset inside X_{01} consisting of the isomorphism classes of all sheaves of the form $\mathcal{O}_C(1)(P_1 + P_2 + P_3 + P_4 - P_5)$, where $C \subset \mathbb{P}^2$ is a smooth quintic curve, $P_1, \ldots, P_5$ are distinct points on C and P_1, P_2, P_3, P_4 are in general linear position.

Proof. We begin by noting that the sheaves giving points in X_{01} are precisely the sheaves $\mathcal{F}$ admitting a resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} q_1 & q_2 & 0 \\ \star & \star & \ell_1 \\ \star & \star & \ell_2 \end{bmatrix},$$

where q_1, q_2 are linearly independent two-forms and ℓ_1, ℓ_2 are linearly independent one-forms. For generic $\mathcal{F}$, q_1 and q_2 have no common linear factor and the conic curves they define intersect in the union Z of four distinct points, no three of which are colinear and also distinct from the common zero of ℓ_1 and ℓ_2. We apply the snake lemma to the exact diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathcal{O}(-1) \arrow[r, "\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}"] \arrow[d] & 2\mathcal{O} \arrow[r] \arrow[d] & \mathcal{I}_x(1) \arrow[r] & 0 \\
 & 0 \arrow[r] & 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \arrow[r, "\varphi"] \arrow[d] & \mathcal{O} \oplus 2\mathcal{O} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(-4) \arrow[r, "\begin{bmatrix} -q_2 \\ q_1 \end{bmatrix}"] & 2\mathcal{O}(-2) \arrow[r, "\begin{bmatrix} q_1 & q_2 \end{bmatrix}"] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

The vertical maps are injections into the second factors, respectively projections onto the first factors. We get the exact sequence

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{I}_x(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

from which the conclusion follows. For the converse we apply the horseshoe lemma to the diagram below. By hypothesis, the morphism $\pi\colon \mathcal{O} \to \mathcal{O}_Z$ lifts to a morphism $\alpha\colon \mathcal{O} \to \mathcal{F}$. Then β, γ, δ are defined in the usual way and we claim that $\delta \neq 0$. If δ were zero, then γ would factor through a morphism $\mathcal{K}er(\pi) \to \mathcal{I}_x(1)$. Since $\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{I}_x(1)) = 0$, this morphism would lift to a map $\eta\colon \mathcal{O} \to \mathcal{I}_x(1)$. The composite map $2\mathcal{O}(-2) \to \mathcal{O} \stackrel{\alpha}{\to} \mathcal{F}$ would then coincide with the composition

$$2\mathcal{O}(-2) \longrightarrow \mathcal{O} \stackrel{-\eta}{\longrightarrow} \mathcal{I}_x(1) \stackrel{\nu}{\longrightarrow} \mathcal{J}_x(1) \stackrel{\xi}{\longrightarrow} \mathcal{F},$$

hence $\alpha + \xi \circ \nu \circ \eta$ would factor through a morphism $\sigma\colon \mathcal{O}_Z \to \mathcal{F}$. We would have $\pi = \zeta \circ \alpha = \zeta \circ \alpha + \zeta \circ \xi \circ \nu \circ \eta = \zeta \circ \sigma \circ \pi$, hence $\zeta \circ \sigma$ would be the identity morphism. The extension would split, contradicting our hypothesis on $\mathcal{F}$. Combining the resolutions for $\mathcal{J}_x(1)$ and $\mathcal{O}_Z$ and cancelling $\mathcal{O}(-4)$ we obtain the resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O} \oplus \mathcal{I}_x(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

From this we easily get a resolution for $\mathcal{F}$ as at the beginning of this proof.

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathcal{O}(-4) \arrow[d] \arrow[dll, "\delta"'] & \\
 & \mathcal{O}(-4) \arrow[d] & & 2\mathcal{O}(-2) \arrow[d] \arrow[dll, "\gamma"'] \arrow[ddll, "\beta"'] & \\
 & \mathcal{I}_x(1) \arrow[d, "\nu"'] & & \mathcal{O} \arrow[d, "\pi"] \arrow[dl, "\alpha"'] & \\
0 \arrow[r] & \mathcal{J}_x(1) \arrow[r, "\xi"] \arrow[d] & \mathcal{F} \arrow[r, "\zeta"] & \mathcal{O}_Z \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}
$$

Assume now that C is smooth and write $Z = \{P_1, P_2, P_3, P_4\}$, $x = P_5$. Clearly, the only non-trivial extension sheaf of $\mathcal{O}_Z$ by $\mathcal{J}_x(1)$ is isomorphic to $\mathcal{F} = \mathcal{O}_C(1)(P_1 + P_2 + P_3 + P_4 - P_5)$. To finish the proof of the proposition we must show that $\mathcal{F}$ has a global section that does not vanish at any point of Z. For $1 \leq i \leq 4$, let $\varepsilon_i \colon \mathrm{H}^0(\mathcal{O}_Z) \to \mathbb{C}$ be the linear form of evaluation at P_i. Let $\delta \colon \mathrm{H}^0(\mathcal{O}_Z) \to \mathrm{H}^1(\mathcal{J}_x(1))$ be the connecting homomorphism in the long exact cohomology sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(1)(-x) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We must show that each ε_i is not orthogonal to $\mathrm{Ker}(\delta)$. This is equivalent to saying that ε_i is not in the image of the dual map δ^*. By Serre duality δ^* is the restriction morphism

$$
\begin{tikzcd}
\mathrm{H}^0(\mathcal{O}_C(-1)(x) \otimes \omega_C) \arrow[r] \arrow[d, equal] & \mathrm{H}^0((\mathcal{O}_C(-1)(x) \otimes \omega_C)|_Z) \ . \arrow[d, equal] \\
\mathrm{H}^0(\mathcal{O}_C(1)(x)) \arrow[d, equal] & \mathrm{H}^0(\mathcal{O}_C(1)(x)|_Z) \arrow[d, equal] \\
\mathrm{H}^0(\mathcal{O}_C(1)) & \mathrm{H}^0(\mathcal{O}_C(1)|_Z)
\end{tikzcd}
$$

The identity $\mathrm{H}^0(\mathcal{O}_C(1)(x)) \simeq \mathrm{H}^0(\mathcal{O}_C(1)) \simeq V^*$ follows from the fact that the connecting homomorphism in the long exact cohomology sequence associated

to the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{O}_C(1)(x) \longrightarrow \mathbb{C}_x \longrightarrow 0$$

is non-zero. Indeed, its dual is the restriction map $\mathrm{H}^0(\mathcal{O}_C(1)) \to \mathrm{H}^0(\mathcal{O}_C(1)|_x)$. This map is clearly non-zero. Now $\delta^*(u)$ is a multiple of ε_i if and only if the linear form u vanishes at P_j for all $j \neq i$. By hypothesis the points P_j, $j \neq i$, are non-colinear, so there is no such form u and we conclude that ε_i is not in the image of δ^*. □

Proposition 2.3.3. *The sheaves $\mathcal{F}$ in X_1 are precisely the non-split extension sheaves of the form*

$$0 \longrightarrow \mathcal{E}^{\mathrm{D}}(1) \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathbb{C}_x$ is the structure sheaf of a point $x \in \mathbb{P}^2$ and $\mathcal{E}$ is in X_2. Here $\mathcal{E}^{\mathrm{D}} = \mathcal{E}xt^1(\mathcal{E}, \omega_{\mathbb{P}^2})$ signifies the dual sheaf of $\mathcal{E}$. Taking into account the duality isomorphism [10]*, the sheaves $\mathcal{E}^{\mathrm{D}}(1)$ are precisely the sheaves $\mathcal{G}$ in the dual stratum $X_2^{\mathrm{D}} \subset \mathrm{M}_{\mathbb{P}^2}(5,2)$ defined by the relations*

$$h^0(\mathcal{G}(-1)) = 0, \qquad h^1(\mathcal{G}) = 1, \qquad h^1(\mathcal{G} \otimes \Omega^1(1)) = 3.$$

The generic sheaves in X_1 are of the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4)$, where $C \subset \mathbb{P}^2$ is a smooth quintic curve, P_i are four distinct points on C and P_1, P_2, P_3 are non-colinear. In particular, X_1 lies in the closure of X_{01}.

Proof. Let $\mathcal{F}$ be in X_1. As in the proof of 2.3.2, the snake lemma gives an exact sequence

$$0 \longrightarrow \mathcal{K}er(\varphi_{11}) \xrightarrow{\alpha} \mathcal{C}oker(\varphi_{22}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}oker(\varphi_{11}) \longrightarrow 0.$$

Because of the form of φ_{11} given at 2.1.2(ii), we have the isomorphisms $\mathcal{K}er(\varphi_{11}) \simeq \mathcal{O}(-3)$ and $\mathcal{C}oker(\varphi_{11}) \simeq \mathbb{C}_x$ for a point $x \in \mathbb{P}^2$. Denoting $\mathcal{G} = \mathcal{C}oker(\alpha)$, we have an extension

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

Again from 2.1.2(ii), we know that φ_{22} is injective, hence $\mathcal{G}$ has a resolution of the form

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \xrightarrow{\psi} 3\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

with $\psi_{12} = \varphi_{22}$. According to the proof of 2.1.3, $\mathcal{G}$ is in the dual stratum X_2^{D}.

Conversely, assume that $\mathcal{F}$ is an extension as in the proposition. Using the horseshoe lemma we combine the resolutions

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \xrightarrow{\psi} 3\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \longrightarrow \mathbb{C}_x \longrightarrow 0$$

to obtain a resolution

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Note that $\mathrm{Ext}^1(\mathbb{C}_x, 3\mathcal{O}) = 0$, so we can use the arguments at 2.3.2 to show that the extension would split if the morphism $\mathcal{O}(-3) \to \mathcal{O}(-3)$ in the above complex were zero. We deduce that this morphism is non-zero, so we may cancel $\mathcal{O}(-3)$ to get a resolution as in 2.1.2(ii).

The part of the proposition concerning generic sheaves follows from the corresponding part of proposition 2.3.4 below.

To see that X_1 is included in $\overline{X}_{01}$ we choose a point in X_1 represented by $\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4)$. We may assume that the line through P_1 and P_2 intersects C at five distinct points P_1, P_2, Q_1, Q_2, Q_3, which are also distinct from P_3 and P_4. Then

$$\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4) \simeq \mathcal{O}_C(1)(Q_1 + Q_2 + Q_3 - P_3 + P_4).$$

Clearly, we can find points R_1, R_2, R_3 on C, converging to Q_1, Q_2, Q_3 respectively, which are distinct from P_3 and such that R_1, R_2, R_3, P_4 are in general linear position. Then $\mathcal{O}_C(1)(R_1 + R_2 + R_3 + P_4 - P_3)$ represents a point in X_{01} converging to the chosen point in X_1. $\square$

We recall from the proof of 2.1.3 that the sheaves $\mathcal{G}$ giving points in the dual stratum $X_2^{\mathrm{D}} \subset \mathrm{M}_{\mathbb{P}^2}(5,2)$ are precisely the sheaves that admit a resolution of the form

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \xrightarrow{\psi} 3\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where ψ_{12} has linearly independent maximal minors. We consider the open subset X_{20}^{D} of X_2^{D} given by the condition that the maximal minors of ψ_{12} have no common linear factor and we denote $X_{21}^{\mathrm{D}} = X_2^{\mathrm{D}} \setminus X_{20}^{\mathrm{D}}$.

Proposition 2.3.4. (i) *The sheaves $\mathcal{G}$ from X_{20}^{D} are precisely the twisted ideal sheaves $\mathcal{J}_Z(2)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 3 not contained in a line, contained in a quintic curve $C \subset \mathbb{P}^2$, and $\mathcal{J}_Z \subset \mathcal{O}_C$ is its ideal sheaf.*

The generic sheaves in X_2^{D} are of the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3)$, where C is a smooth quintic curve and P_1, P_2, P_3 are non-colinear points on C.

By duality, the generic sheaves in X_2 are of the form $\mathcal{O}_C(1)(P_1 + P_2 + P_3)$. In particular, X_2 lies in the closure of X_1.

(ii) *The sheaves $\mathcal{G}$ from X_{21}^{D} are precisely the extension sheaves of the form*

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_C(1) \longrightarrow 0$$

where $L \subset \mathbb{P}^2$ is a line, $C \subset \mathbb{P}^2$ is a quartic curve and such that the image of $\mathcal{G}$ under the canonical map

$$\mathrm{Ext}^1(\mathcal{O}_C(1), \mathcal{O}_L(-1)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}_L(-1))$$

is non-zero.

Proof. (i) According to 4.5 and 4.6 [2], $\mathcal{C}oker(\psi_{12}) \simeq \mathcal{I}_Z(2)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 3 not contained in a line and $\mathcal{I}_Z \subset \mathcal{O}$ is its ideal sheaf. Conversely, every $\mathcal{I}_Z(2)$ is the cokernel of some morphism

$\psi_{12}\colon 2\mathcal{O}(-1)\to 3\mathcal{O}$ whose maximal minors are linearly independent and have no common linear factor. Thus, the sheaves $\mathcal{G}\in X_{20}^{\mathrm{D}}$ are precisely the cokernels of injective morphisms $\mathcal{O}(-3)\to\mathcal{I}_Z(2)$. If C is the quintic curve defined by the inclusion $\mathcal{O}(-3)\subset\mathcal{I}_Z(2)\subset\mathcal{O}(2)$, then it is easy to see that $\mathcal{G}\simeq\mathcal{J}_Z(2)$.

To see that X_2 is included in $\overline{X}_1$ we choose a generic sheaf in X_2 of the form $\mathcal{O}_C(1)(P_1+P_2+P_3)$. We may assume that the line through P_1 and P_2 intersects C at five distinct points P_1,P_2,Q_1,Q_2,Q_3. For non-colinear points R_1,R_2,R_3 on C, converging to Q_1,Q_2,Q_3 respectively, the sheaf

$$\mathcal{O}_C(2)(-R_1-R_2-R_3+P_3)\simeq\mathcal{O}_C(1)(P_1+P_2+P_3+Q_1+Q_2+Q_3-R_1-R_2-R_3)$$

represents a point in X_1 converging to the point given by $\mathcal{O}_C(1)(P_1+P_2+P_3)$.

(ii) Let ℓ be a common linear factor of the maximal minors of ψ_{12}. Consider the line L with equation $\ell=0$. According to 3.3.3 [4], $\mathcal{C}oker(\psi_{12})\simeq\mathcal{E}_L$, where $\mathcal{E}_L$ is the unique non-split extension

$$0\longrightarrow\mathcal{O}_L(-1)\longrightarrow\mathcal{E}_L\longrightarrow\mathcal{O}(1)\longrightarrow 0.$$

Conversely, every $\mathcal{E}_L$ is the cokernel of some morphism $\psi_{12}\colon 2\mathcal{O}(-1)\to 3\mathcal{O}$ with linearly independent maximal minors which have a common linear factor. Thus, the sheaves $\mathcal{G}$ giving points in X_{21}^{D} are precisely the cokernels of the injective morphisms $\mathcal{O}(-3)\to\mathcal{E}_L$. Let $C\subset\mathbb{P}^2$ be the quartic curve defined by the composition $\mathcal{O}(-3)\to\mathcal{E}_L\to\mathcal{O}(1)$. We apply the snake lemma to the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-3) \arrow[r] \arrow[d, equal] & \mathcal{E}_L \arrow[r] \arrow[d, "\alpha"] & \mathcal{G} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(-3) \arrow[r] & \mathcal{O}(1) \arrow[r] & \mathcal{O}_C(1) \arrow[r] & 0
\end{tikzcd}$$

As $\mathcal{K}er(\alpha)\simeq\mathcal{O}_L(-1)$, we obtain an extension

$$0\longrightarrow\mathcal{O}_L(-1)\longrightarrow\mathcal{G}\longrightarrow\mathcal{O}_C(1)\longrightarrow 0$$

which maps to the class of $\mathcal{E}_L$ in $\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}(1),\mathcal{O}_L(-1)))$. The converse is clear, in view of the fact that $\mathrm{Ext}^1(\mathcal{O}(1),\mathcal{O}_L(-1))\simeq\mathbb{C}$. $\square$

Proposition 2.3.5. *The sheaves $\mathcal{F}$ giving points in X_3 are precisely the twisted ideal sheaves $\mathcal{J}_Z(2)$, where $Z\subset\mathbb{P}^2$ is a zero-dimensional scheme of length 2 contained in a quintic curve C and $\mathcal{J}_Z\subset\mathcal{O}_C$ is its ideal sheaf.*

The generic sheaves in X_3 are of the form $\mathcal{O}_C(1)(P_1+P_2+P_3)$, where $C\subset\mathbb{P}^2$ is a smooth quintic curve and P_1,P_2,P_3 are distinct colinear points on C. In particular, X_3 lies in the closure of X_2.

Proof. Adopting the notations of 2.1.4, we notice that the restriction of φ to $\mathcal{O}(-1)$ has cokernel $\mathcal{I}_Z(2)$, where Z is the intersection of the line with equation $\varphi_{12}=0$ and the conic with equation $\varphi_{22}=0$. Thus the sheaves $\mathcal{F}$ in X_3 are precisely the cokernels of injective morphisms $\mathcal{O}(-3)\to\mathcal{I}_Z(2)$. Let C

be the quintic curve defined by the inclusion $\mathcal{O}(-3) \subset \mathcal{I}_Z(2) \subset \mathcal{O}(2)$. Clearly $\mathcal{F} \simeq \mathcal{J}_Z(2)$.

To see that $X_3 \subset \overline{X}_2$ choose a generic sheaf $\mathcal{O}_C(1)(P_1 + P_2 + P_3)$ in X_3. Clearly, we can find non-colinear points Q_1, Q_2, Q_3 on C converging to P_1, P_2, P_3 respectively. Then $\mathcal{O}_C(1)(Q_1 + Q_2 + Q_3)$ represents a point in X_2 converging to the chosen point in X_3. $\square$

From what was said above we can summarise:

Proposition 2.3.6. $\{X_0 \setminus X_{01}, X_{01}, X_1, X_2, X_3\}$ *represents a stratification of* $\mathrm{M}_{\mathbb{P}^2}(5,3)$ *by locally closed irreducible subvarieties of codimensions* $0, 1, 2, 3, 4$.

3. Euler characteristic one or four

3.1. **Locally free resolutions for semi-stable sheaves.**

Proposition 3.1.1. *Every sheaf* $\mathcal{F}$ *giving a point in* $\mathrm{M}_{\mathbb{P}^2}(5,1)$ *and satisfying the condition* $h^1(\mathcal{F}) = 0$ *also satisfies the condition* $h^0(\mathcal{F}(-1)) = 0$. *These sheaves are precisely the sheaves with resolution*

$$0 \longrightarrow 4\mathcal{O}(-2) \xrightarrow{\varphi} 3\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{11} *is not equivalent to a morphism represented by a matrix of the form*

$$\left[\begin{array}{cc} \psi & 0 \\ \star & \star \end{array}\right], \quad \textit{with} \quad \psi \colon m\mathcal{O}(-2) \longrightarrow m\mathcal{O}(-1), \quad m = 1, 2, 3.$$

Proof. According to 4.2 [9], every sheaf $\mathcal{G}$ giving a point in $\mathrm{M}_{\mathbb{P}^2}(5,4)$ and satisfying the condition $h^0(\mathcal{G}(-1)) = 0$ also satisfies the condition $h^1(\mathcal{G}) = 0$ and has a resolution

$$0 \longrightarrow \mathcal{O}(-2) \oplus 3\mathcal{O}(-1) \xrightarrow{\varphi} 4\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where φ_{12} is not equivalent to a morphism represented by a matrix of the form

$$\left[\begin{array}{cc} \star & \psi \\ \star & 0 \end{array}\right], \quad \text{with} \quad \psi \colon m\mathcal{O}(-1) \longrightarrow m\mathcal{O}, \quad m = 1, 2, 3.$$

The result follows by duality. $\square$

Proposition 3.1.2. *Let* $\mathcal{F}$ *be a sheaf giving a point in* $\mathrm{M}_{\mathbb{P}^2}(5,1)$ *satisfying the conditions* $h^1(\mathcal{F}) = 1$ *and* $h^0(\mathcal{F}(-1)) = 0$. *Then* $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$ *or* 1. *The sheaves in the first case are precisely the sheaves that have a resolution of the form*

$$\text{(i)} \qquad 0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{12} *and* φ_{22} *are linearly independent two-forms. The sheaves from the second case are precisely the sheaves with resolution*

$$\text{(ii)} \qquad 0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} q & \ell & 0 \\ \varphi_{21} & \varphi_{22} & \ell_1 \\ \varphi_{31} & \varphi_{32} & \ell_2 \end{bmatrix},$$

where ℓ is non-zero, q is non-divisible by ℓ and ℓ_1, ℓ_2 are linearly independent one-forms.

Proof. Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(5,1)$ and satisfy the conditions $h^1(\mathcal{F}) = 1$ and $h^0(\mathcal{F}(-1)) = 0$. Put $m = h^0(\mathcal{F} \otimes \Omega^1(1))$. The Beilinson free monad (2.2.1) [4] for $\mathcal{F}$ reads

$$0 \longrightarrow 4\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow (m+3)\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{O} \longrightarrow 0$$

and gives the resolution

$$0 \longrightarrow 4\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow \Omega^1 \oplus m\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Combining this with the standard resolution for Ω^1 we obtain the following exact sequence:

$$0 \longrightarrow \mathcal{O}(-3) \oplus 4\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

with $\varphi_{13} = 0$, $\varphi_{23} = 0$. As in the proof of 2.1.4, we have $\operatorname{rank}(\varphi_{12}) = 3$. Canceling $3\mathcal{O}(-2)$ we get the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} m\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

with $\varphi_{13} = 0$. From the injectivity of φ we must have $m \le 2$. If $m = 2$, then $\mathcal{C}oker(\varphi_{23})$ is a destabilising subsheaf of $\mathcal{F}$. We conclude that $m = 0$ or 1.

Assume that $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$. We arrive at resolution (i). If φ_{12} and φ_{22} were linearly dependent, then $\mathcal{F}$ would have a destabilising subsheaf of the form $\mathcal{O}_C$, for a conic curve $C \subset \mathbb{P}^2$. Conversely, we assume that $\mathcal{F}$ has resolution (i) and we must show that $\mathcal{F}$ cannot have a destabilising subsheaf $\mathcal{E}$. We may restrict our attention to semi-stable sheaves $\mathcal{E}$. As $\mathcal{F}$ is generated by global sections, we must have $h^0(\mathcal{E}) < h^0(\mathcal{F}) = 2$. Thus $\mathcal{E}$ is in $\mathrm{M}_{\mathbb{P}^2}(r,1)$ for some $1 \le r \le 4$ and we have $h^1(\mathcal{E}) = 0$. Moreover, $\mathrm{H}^0(\mathcal{E} \otimes \Omega^1(1))$ vanishes because the corresponding cohomology group for $\mathcal{F}$ vanishes. This excludes the possibility $r = 1$. In the case $r = 2$, $\mathcal{E}$ is the structure sheaf of a conic curve, but this, by virtue of our hypothesis on φ_{12} and φ_{22}, is not allowed. If $\mathcal{E}$ is in $\mathrm{M}_{\mathbb{P}^2}(3,1)$, then, according to [8], $\mathcal{E}$ has resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0.$$

If $\mathcal{E}$ is in $\mathrm{M}_{\mathbb{P}^2}(4,1)$, then, from the description of this moduli space found in [4], we see that $\mathcal{E}$ has resolution

$$0 \longrightarrow 3\mathcal{O}(-2) \longrightarrow 2\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0.$$

It is easy to see that the first exact sequence must fit into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 2\mathcal{O}(-2) \arrow[r, "\psi"] \arrow[d, "\beta"] & \mathcal{O}(-1) \oplus \mathcal{O} \arrow[r] \arrow[d, "\alpha"] & \mathcal{E} \arrow[r] \arrow[d] & 0\, . \\
0 \arrow[r] & \mathcal{O}(-3) \oplus \mathcal{O}(-2) \arrow[r, "\varphi"] & 2\mathcal{O} \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

From the fact that α and $\alpha(1)$ are injective on global sections we see that $\mathcal{C}oker(\alpha)$ is supported on a line. This is impossible because $\mathcal{O}(-3)$ maps injectively to $\mathcal{C}oker(\beta)$ which maps injectively to $\mathcal{C}oker(\alpha)$. The same argument applies to the second exact sequence as well, except that $\mathcal{C}oker(\alpha)$ this time would be supported on a point.

Assume now that $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$. We arrive at resolution (ii). If ℓ_1, ℓ_2 were linearly dependent, then $\mathcal{F}$ would have a destabilising subsheaf of the form $\mathcal{O}_L$, for a line $L \subset \mathbb{P}^2$. If $\ell = 0$, then $\mathcal{F}$ would have a destabilising quotient sheaf of the form $\mathcal{O}_C(-1)$, for a conic curve $C \subset \mathbb{P}^2$. If ℓ divided q, then $\mathcal{F}$ would have a destabilising quotient sheaf of the form $\mathcal{O}_L(-1)$. Conversely, we assume that $\mathcal{F}$ has resolution (ii) and we must show that there is no destabilising subsheaf. Let x be the point with equations $\ell_1 = 0$, $\ell_2 = 0$ and let $Z \subset \mathbb{P}^2$ be the zero-dimensional subscheme of length 2 given by the equations $\ell = 0$, $q = 0$. We apply the snake lemma to the exact diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathcal{O}(-1) \arrow[r, "\varphi_{23}"] \arrow[d] & 2\mathcal{O} \arrow[r] \arrow[d] & \mathcal{I}_x(1) \arrow[r] & 0 \\
 & 0 \arrow[r] & \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \arrow[r, "\varphi"] \arrow[d] & \mathcal{O}(-1) \oplus 2\mathcal{O} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(-4) \arrow[r, "\left[\begin{smallmatrix} -\ell \\ q \end{smallmatrix}\right]"] & \mathcal{O}(-3) \oplus \mathcal{O}(-2) \arrow[r, "\left[\begin{smallmatrix} q & \ell \end{smallmatrix}\right]"] \arrow[d] & \mathcal{O}(-1) \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

We get the exact sequence

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{I}_x(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

Let C be the quintic curve defined by the inclusion $\mathcal{O}(-4) \subset \mathcal{I}_x(1) \subset \mathcal{O}(1)$. We obtain an exact sequence:

$$0 \longrightarrow \mathcal{J}_x(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

where $\mathcal{J}_x \subset \mathcal{O}_C$ is the ideal sheaf of x on C. Let $\mathcal{F}' \subset \mathcal{F}$ be a non-zero subsheaf of multiplicity at most 4. Denote by $\mathcal{C}'$ its image in $\mathcal{O}_Z$ and put $\mathcal{K} = \mathcal{F}' \cap \mathcal{J}_x(1)$. By [9], lemma 6.7, there is a sheaf $\mathcal{A} \subset \mathcal{O}_C(1)$ containing $\mathcal{K}$ such that $\mathcal{A}/\mathcal{K}$ is supported on finitely many points and $\mathcal{O}_C(1)/\mathcal{A} \simeq \mathcal{O}_S(1)$ for a curve $S \subset \mathbb{P}^2$ of degree $d \le 4$. The slope of $\mathcal{F}'$ can be estimated as follows:

$$\begin{aligned} P_{\mathcal{F}'}(t) &= P_{\mathcal{K}}(t) + h^0(\mathcal{C}') \\ &= P_{\mathcal{A}}(t) - h^0(\mathcal{A}/\mathcal{K}) + h^0(\mathcal{C}') \\ &= P_{\mathcal{O}_C}(t+1) - P_{\mathcal{O}_S}(t+1) - h^0(\mathcal{A}/\mathcal{K}) + h^0(\mathcal{C}') \\ &= (5-d)t + \frac{d^2-5d}{2} - h^0(\mathcal{A}/\mathcal{K}) + h^0(\mathcal{C}'), \\ p(\mathcal{F}') &= -\frac{d}{2} + \frac{h^0(\mathcal{C}') - h^0(\mathcal{A}/\mathcal{K})}{5-d} \le -\frac{d}{2} + \frac{2}{5-d} < \frac{1}{5} = p(\mathcal{F}). \end{aligned}$$

We conclude that $\mathcal{F}$ is semi-stable. $\square$

Proposition 3.1.3. *There are no sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(5,1)$ and satisfying the conditions $h^0(\mathcal{F}(-1)) = 0$ and $h^1(\mathcal{F}) = 2$.*

Proof. By duality, we need to show that there are no sheaves $\mathcal{G}$ in $\mathrm{M}_{\mathbb{P}^2}(5,4)$ satisfying the conditions $h^0(\mathcal{G}(-1)) = 2$ and $h^1(\mathcal{G}) = 0$. Assume that there is such a sheaf $\mathcal{G}$. Write $m = h^1(\mathcal{G} \otimes \Omega^1(1))$. The Beilinson monad gives a resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\eta} 3\mathcal{O}(-2) \oplus (m+3)\mathcal{O}(-1) \xrightarrow{\varphi} m\mathcal{O}(-1) \oplus 4\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

$$\eta = \begin{bmatrix} 0 \\ \psi \end{bmatrix}.$$

Here $\varphi_{12} = 0$. As $\mathcal{G}$ maps surjectively onto $\mathcal{C}oker(\varphi_{11})$, the latter has rank zero, forcing $m \le 3$. In the case $m = 3$, $\mathcal{C}oker(\varphi_{11})$ has Hilbert polynomial $P(t) = 3t$, so the semi-stability of $\mathcal{G}$ gets contradicted. Thus $m \le 2$.

We claim that any matrix representing a morphism equivalent to ψ has three linearly independent entries on each column. The argument uses the fact that $\mathcal{G}$ has no zero-dimensional torsion and is analogous to the proof that the vector space H from 2.1.4 has dimension 3. Thus we may assume that one of the columns of ψ is

$$\begin{bmatrix} 0 & \cdots & 0 & X & Y & Z \end{bmatrix}^{\mathrm{T}}.$$

Let φ_0 be the matrix made of the last three columns of φ_{22}. The rows of φ_0 are linear combinations of the rows of the matrix

$$\begin{bmatrix} -Y & X & 0 \\ -Z & 0 & X \\ 0 & -Z & Y \end{bmatrix}.$$

It is easy to see that the elements on any row of φ_0 are linearly dependent. The rows of φ_0 cannot span a vector space of dimension 1, otherwise φ_{22} would be equivalent to a morphism represented by a matrix having a zero-column, hence $\mathcal{O}(-1) \subset \mathcal{K}er(\varphi)$, which is absurd. $\mathcal{K}er(\varphi_0)$ is isomorphic to $\mathcal{O}(-2)$ because φ_0 has at least two linearly independent rows. This excludes the case $m = 0$ because in that case $\varphi_0 = \varphi_{22}$ and $\mathcal{K}er(\varphi_{22}) \simeq 2\mathcal{O}(-2)$. In the remaining two cases we shall prove that the rows of φ_0 cannot span a vector space of dimension 2. We argue by contradiction. Assume that $m = 2$ and that φ_0 is equivalent to a matrix of the form

$$\left[\begin{array}{c} 0 \\ \xi \end{array} \right],$$

where ξ is a 2×3-matrix with linearly independent rows. Then $\mathcal{K}er(\xi) \simeq \mathcal{O}(-2)$ and $\mathcal{C}oker(\xi) \simeq \mathcal{O}_L(1)$ for a line $L \subset \mathbb{P}^2$. The first isomorphism is obvious and tells us that the maximal minors of ξ are linearly independent and have a common linear factor, say ℓ. Let $L \subset \mathbb{P}^2$ be the line with equation $\ell = 0$. $\mathcal{C}oker(\xi)$ is supported on L and has Hilbert polynomial $P(t) = t + 2$. Moreover, it is easy to see that ξ has rank 1 at every point of L, hence $\mathcal{C}oker(\xi)$ has no zero-dimensional torsion. This proves the second isomorphism. We now use the argument from the proof of 2.1.4. There is a commutative diagram

$$\begin{tikzcd}
3\mathcal{O}(-1) \arrow[r, "\xi"] \arrow[d] & 2\mathcal{O} \arrow[r] \arrow[d] & \mathcal{O}_L(1) \arrow[r] \arrow[d] & 0 \\
3\mathcal{O}(-2) \oplus 5\mathcal{O}(-1) \arrow[r, "\varphi"] & 2\mathcal{O}(-1) \oplus 4\mathcal{O} \arrow[r] & \mathcal{G} \arrow[r] & 0
\end{tikzcd}$$

in which the first two vertical maps are injective. The induced morphism $\mathcal{O}_L(1) \to \mathcal{G}$ is zero because both sheaves are stable and $p(\mathcal{O}_L(1)) > p(\mathcal{G})$. Thus the map $4\mathcal{O} \to \mathcal{G}$ is not injective on global sections. On the other hand, $\mathrm{H}^0(\mathcal{C}oker(\eta))$ vanishes, hence the map $4\mathcal{O} \to \mathcal{G}$ is injective on global sections. We have arrived at a contradiction. We conclude that the rows of φ_0 span a vector space of dimension 3.

Modulo elementary operations on rows and columns, ψ is equivalent to a morphism represented by a matrix having one of the following forms:

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ X & R \\ Y & S \\ Z & T \end{array} \right] \qquad \text{or} \qquad \left[\begin{array}{cc} 0 & 0 \\ X & 0 \\ Y & R \\ Z & S \\ 0 & T \end{array} \right] \qquad \text{or} \qquad \left[\begin{array}{cc} X & 0 \\ Y & 0 \\ Z & R \\ 0 & S \\ 0 & T \end{array} \right].$$

Here R, S, T form a basis of V^*. In the first case the triple (R, S, T) is a multiple of (X, Y, Z), because, as we saw above, $\mathcal{K}er(\varphi_0) \simeq \mathcal{O}(-2)$. Thus ψ is represented by a matrix with a zero-column. This is absurd. In the second

case we can perform elementary row operations on the matrix

$$\begin{bmatrix} X & 0 \\ 0 & X \\ -Z & Y \end{bmatrix} \quad \text{to get the matrix} \quad \begin{bmatrix} -S & R \\ -T & 0 \\ 0 & -T \end{bmatrix}.$$

It follows that

$$\mathrm{span}\{X\} = \mathrm{span}\{X, Z\} \cap \mathrm{span}\{X, Y\} = \mathrm{span}\{S, T\} \cap \mathrm{span}\{R, T\} = \mathrm{span}\{T\}$$

and $(-S, R) = a(-Z, Y) + (bX, cX)$ for some $a, b, c \in \mathbb{C}$. Thus ψ is equivalent to the morphism represented by the matrix

$$\begin{bmatrix} 0 & X & Y & Z & 0 \\ 0 & 0 & 0 & 0 & X \end{bmatrix}^{\mathrm{T}}.$$

This, as we saw above, is not possible. In the third case we can perform elementary row operations on the matrix

$$\begin{bmatrix} 0 \\ X \\ Y \end{bmatrix} \quad \text{to get the matrix} \quad \begin{bmatrix} S \\ T \\ 0 \end{bmatrix}.$$

Thus, we may assume that $S = X$, $T = Y$, $R = Z$. Performing elementary row and column operations on ψ we can get a matrix with three zeros on a column. This, as we saw above, is not possible. Thus far we have eliminated the case when $m = 2$. The case when $m = 1$ can be eliminated in an analogous fashion. We conclude that there are no sheaves $\mathcal{G}$ as above. □

Proposition 3.1.4. *There are no sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(5,1)$ and satisfying the conditions $h^0(\mathcal{F}(-1)) = 0$ and $h^1(\mathcal{F}) \geq 2$.*

Proof. The argument is the same as at proposition 2.1.6 or at 3.2.3 [4]. Using the Beilinson monad for $\mathcal{F}(-1)$ we see that the open subset of $\mathrm{M}_{\mathbb{P}^2}(5,1)$ given by the condition $h^0(\mathcal{F}(-1)) = 0$ is parametrised by an open subset M inside the space of monads of the form

$$0 \longrightarrow 9\mathcal{O}(-1) \xrightarrow{A} 13\mathcal{O} \xrightarrow{B} 4\mathcal{O}(1) \longrightarrow 0.$$

The map $\Phi \colon M \to \mathrm{Hom}(13\mathcal{O}, 4\mathcal{O}(1))$ is defined by $\Phi(A, B) = B$. Using the vanishing of $\mathrm{H}^1(\mathcal{F}(1))$ for an arbitrary sheaf in $\mathrm{M}_{\mathbb{P}^2}(5,1)$, we prove that Φ has surjective differential at every point of M. This further leads to the conclusion that the set of monads in M whose cohomology sheaf $\mathcal{F}$ satisfies $h^1(\mathcal{F}) \geq 2$ is included in the closure of the set of monads for which $h^1(\mathcal{F}) = 2$. According to 3.1.3, the latter set is empty, hence the former set is empty, too. □

Proposition 3.1.5. *The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(5,1)$ and satisfying the condition $h^0(\mathcal{F}(-1)) > 0$ are precisely the sheaves with resolution of the form*

$$0 \longrightarrow 2\mathcal{O}(-3) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} \ell_1 & \ell_2 \\ f_1 & f_2 \end{bmatrix},$$

where ℓ_1, ℓ_2 *are linearly independent one-forms. For these sheaves we have* $h^0(\mathcal{F}(-1)) = 1$ *and* $h^1(\mathcal{F}) = 2$. *These sheaves are precisely the non-split extension sheaves of the form*

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $C \subset \mathbb{P}^2$ *is a quintic curve and* $\mathbb{C}_x$ *is the structure sheaf of a point.*

Proof. Assume that $\mathcal{F}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,1)$ and satisfies $h^0(\mathcal{F}(-1)) > 0$. As in the proof of 2.1.3 [4], there is an injective morphism $\mathcal{O}_C \to \mathcal{F}(-1)$ for some quintic curve $C \subset \mathbb{P}^2$. We obtain a non-split extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

Conversely, using the fact that $\mathcal{O}_C$ is stable, it is easy to see that any non-split extension sheaf as above gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,1)$.

Assume now that $\mathcal{F}$ has a resolution as in the proposition. Let x be the point given by the equations $\ell_1 = 0$, $\ell_2 = 0$ and let $\mathcal{I}_x \subset \mathcal{O}$ be its ideal sheaf. Let $f = \ell_1 f_2 - \ell_2 f_1$ and let C be the quintic curve with equation $f = 0$. We apply the snake lemma to the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-4) \arrow[r, "\begin{bmatrix} -\ell_2 \\ \ell_1 \end{bmatrix}"] \arrow[d, "f"] & 2\mathcal{O}(-3) \arrow[r] \arrow[d, "\varphi"] & \mathcal{I}_x(-2) \arrow[r] \arrow[d] & 0\,. \\
0 \arrow[r] & \mathcal{O}(1) \arrow[r, "i"] & \mathcal{O}(-2) \oplus \mathcal{O}(1) \arrow[r, "p"] & \mathcal{O}(-2) \arrow[r] & 0
\end{tikzcd}$$

Here i is the inclusion into the second factor and p is the projection onto the first factor. We deduce that $\mathcal{F}$ is an extension of $\mathbb{C}_x$ by $\mathcal{O}_C(1)$. As $h^0(\mathcal{F}) = 3$, the extension does not split.

Conversely, assume that $\mathcal{F}$ is a non-split extension of $\mathbb{C}_x$ by $\mathcal{O}_C(1)$. We construct a resolution of $\mathcal{F}$ from the standard resolution of $\mathcal{O}_C(1)$ and from the resolution

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

using the horseshoe lemma. We obtain a resolution of the form

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-3) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

If the map $\mathcal{O}(-4) \to \mathcal{O}(-4)$ in the above resolution were zero, then, as in the proof of 2.3.2, the extension would split. This would be contrary to our hypothesis. We conclude that $\mathcal{O}(-4)$ can be cancelled in the above exact sequence and we arrive at the resolution from the proposition. $\square$

3.2. **Description of the strata as quotients.** In subsection 3.1 we found that the moduli space $\mathrm{M}_{\mathbb{P}^2}(5,1)$ can be decomposed into four strata:

- an open stratum X_0 given by the condition $h^1(\mathcal{F}) = 0$;
- a locally closed stratum X_1 of codimension 2 given by the conditions $h^0(\mathcal{F}(-1)) = 0$, $h^1(\mathcal{F}) = 1$, $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$;
- a locally closed stratum X_2 of codimension 3 given by the conditions $h^0(\mathcal{F}(-1)) = 0$, $h^1(\mathcal{F}) = 1$, $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$;
- the stratum X_3 of codimension 5 given by the conditions $h^0(\mathcal{F}(-1)) = 1$, $h^1(\mathcal{F}) = 2$. We shall see below at 3.2.5 that X_3 is closed.

In the sequel X_i will be equipped with the canonical reduced structure induced from $\mathrm{M}_{\mathbb{P}^2}(5,1)$. Let W_0, W_1, W_2, W_3 be the sets of morphisms φ from 3.1.1, 3.1.2(i), 3.1.2(ii), respectively 3.1.5. Each sheaf $\mathcal{F}$ giving a point in X_i is the cokernel of a morphism $\varphi \in W_i$. Let $\mathbb{W}_i$ be the ambient vector spaces of morphisms of sheaves containing W_i, e.g. $\mathbb{W}_0 = \mathrm{Hom}(4\mathcal{O}(-2), 3\mathcal{O}(-1) \oplus \mathcal{O})$. Let G_i be the natural groups of automorphisms acting by conjugation on $\mathbb{W}_i$. In this subsection we shall prove that there exist geometric quotients W_i/G_i, which are smooth quasiprojective varieties, such that $W_i/G_i \simeq X_i$. We shall also give concrete descriptions of these quotients.

Proposition 3.2.1. *There exists a geometric quotient W_0/G_0, which is a proper open subset inside a fibre bundle over $\mathrm{N}(3,4,3)$ with fibre $\mathbb{P}^{14}$. Moreover, W_0/G_0 is isomorphic to X_0.*

Proof. The situation is analogous to 2.2.4. Let $\Lambda = (\lambda_1, \mu_1, \mu_2)$ be a polarisation for the action of G_0 on $\mathbb{W}_0$ satisfying $0 < \mu_2 < 1/4$. W_0 is the proper open invariant subset of injective morphisms inside $\mathbb{W}_0^{ss}(\Lambda)$. Let $\mathrm{N}(3,4,3)$ be the moduli space of semi-stable Kronecker modules $f\colon 4\mathcal{O}(-2) \to 3\mathcal{O}(-1)$ and let

$$\theta\colon p_1^*(E) \otimes p_2^*(\mathcal{O}(-2)) \longrightarrow p_1^*(F) \otimes p_2^*(\mathcal{O}(-1))$$

be the morphism of sheaves on $\mathrm{N}(3,4,3) \times \mathbb{P}^2$ induced from the universal morphism τ. Then $\mathcal{U} = p_{1*}(\mathcal{C}oker(\theta^*))$ is a vector bundle of rank 15 on $\mathrm{N}(3,4,3)$ and $\mathbb{P}(\mathcal{U})$ is the geometric quotient $\mathbb{W}_0^{ss}(\Lambda)/G_0$. Thus W_0/G_0 exists and is a proper open subset of $\mathbb{P}(\mathcal{U})$.

The canonical morphism $W_0/G_0 \to X_0$ is bijective and, since X_0 is smooth, it is an isomorphism. $\square$

Proposition 3.2.2. *There exists a geometric quotient W_1/G_1 and it is a proper open subset inside a fibre bundle with fibre $\mathbb{P}^{16}$ and base the Grassmann variety $\mathrm{Grass}(2, S^2V^*)$. Moreover, W_1/G_1 is isomorphic to X_1.*

Proof. The existence of W_1/G_1 follows from 9.3 [5]. Let $\Lambda = (\lambda_1, \lambda_2, \mu_1)$ be a polarisation for the action of G_1 on $\mathbb{W}_1$ satisfying $0 < \lambda_1 < 1/2$. Then $\mathbb{W}_1^{ss}(\Lambda)$ is given by the conditions that φ_{12}, φ_{22} be linearly independent two-forms and that the first column of φ be not a multiple of the second column.

W_1 is the proper open invariant subset of injective morphisms inside $\mathbb{W}_1^{ss}(\Lambda)$. The semi-stable morphisms that are not injective are represented by matrices of the form

$$\begin{bmatrix} q\ell_1 & \ell\ell_1 \\ q\ell_2 & \ell\ell_2 \end{bmatrix}$$

with $\ell \in V^*$ non-zero, $q \in S^2V^*$ non-divisible by ℓ and $\ell_1, \ell_2 \in V^*$ linearly independent. The moduli space $\mathrm{N}(6,1,2)$ of semi-stable Kronecker modules $f\colon \mathcal{O}(-2) \to 2\mathcal{O}$ is isomorphic to $\mathrm{Grass}(2, S^2V^*)$. Let

$$\theta\colon p_1^*(E) \otimes p_2^*(\mathcal{O}(-2)) \longrightarrow p_1^*(F)$$

be the morphism of sheaves on $\mathrm{N}(6,1,2) \times \mathbb{P}^2$ induced from the universal morphism τ. Then $\mathcal{U} = p_{1*}(\mathcal{C}oker(\theta) \otimes p_2^*(\mathcal{O}(3)))$ is a vector bundle of rank 17 over $\mathrm{N}(6,1,2)$ and $\mathbb{P}(\mathcal{U})$ is the geometric quotient $\mathbb{W}_1^{ss}(\Lambda)/G_1$. Thus W_1/G_1 exists and is a proper open subset of the projective variety $\mathbb{P}(\mathcal{U})$.

To show that the canonical bijective morphism $W_1/G_1 \to X_1$ is an isomorphism we shall construct resolution 3.1.2(i) for a sheaf $\mathcal{F}$ giving a point in X_1 in a natural manner from the Beilinson diagram (2.2.3) [4] for $\mathcal{F}$, which has the form

$$\begin{array}{ccccc} 4\mathcal{O}(-2) & \xrightarrow{\varphi_1} & 3\mathcal{O}(-1) & \xrightarrow{\varphi_2} & \mathcal{O} \\ & & & & \\ 0 & & 0 & & 2\mathcal{O} \end{array} .$$

According to 2.2 [4], φ_2 is surjective, so $\mathcal{K}er(\varphi_2) \simeq \Omega^1$. Recall the morphism ρ introduced in the proof of 2.2.4. There is a morphism $\alpha\colon 4\mathcal{O}(-2) \to 3\mathcal{O}(-2)$ such that $\rho \circ \alpha = \varphi_1$. As at 2.2.4, we have $\mathrm{rank}(\alpha) = 3$, forcing

$$\mathcal{K}er(\varphi_2) = \mathcal{I}m(\varphi_1) \quad \text{and} \quad \mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3) \oplus \mathcal{O}(-2).$$

The exact sequence (2.2.5) [4] takes the form

$$0 \longrightarrow \mathcal{K}er(\varphi_1) \xrightarrow{\varphi_5} 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

and gives us resolution 3.1.2(i). In this fashion we construct a local inverse to the morphism $W_1/G_1 \to X_1$. We conclude that this is an isomorphism. $\square$

Proposition 3.2.3. *There exists a geometric quotient W_2/G_2 and it is a proper open subset inside a fibre bundle with fibre $\mathbb{P}^{17}$ and base $Y \times \mathbb{P}^2$, where Y is the Hilbert scheme of zero-dimensional subschemes of $\mathbb{P}^2$ of length 2.*

Proof. To obtain W_2/G_2 we shall construct successively quotients modulo subgroups of G_2, as at 2.2.2 and 2.2.5. Let $W_2' \subset \mathbb{W}_2$ be the locally closed subset of morphisms φ satisfying the conditions from proposition 3.1.2(ii), except injectivity. The pairs of morphisms $(\varphi_{11}, \varphi_{12})$ form an open subset $U_1 \subset \mathrm{Hom}(\mathcal{O}(-3) \oplus \mathcal{O}(-2), \mathcal{O}(-1))$ and the morphisms φ_{23} form an open subset U_2 inside $\mathrm{Hom}(\mathcal{O}(-1), 2\mathcal{O})$. We denote $U = U_1 \times U_2$. W_2' is the trivial

vector bundle on U with fibre $\operatorname{Hom}(\mathcal{O}(-3) \oplus \mathcal{O}(-2), 2\mathcal{O})$. We represent the elements of G_2 by pairs of matrices

$$(g, h) \in \operatorname{Aut}(\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \operatorname{Aut}(\mathcal{O}(-1) \oplus 2\mathcal{O}),$$

$$g = \begin{bmatrix} g_{11} & 0 & 0 \\ u_{21} & g_{22} & 0 \\ u_{31} & u_{32} & g_{33} \end{bmatrix}, \qquad h = \begin{bmatrix} h_{11} & 0 & 0 \\ v_{21} & h_{22} & h_{23} \\ v_{31} & h_{32} & h_{33} \end{bmatrix}.$$

Inside G_2 we distinguish four subgroups: a reductive subgroup $G_{2\mathrm{red}}$ given by the conditions $u_{ij} = 0$, $v_{ij} = 0$, the subgroup S of pairs (g, h) of the form

$$g = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}, \qquad h = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix},$$

with $a, b \in \mathbb{C}^*$, and two unitary subgroups G_2' and G_2''. Here G_2' consists of pairs (g, h) of morphisms of the form

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_{31} & u_{32} & 1 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 & 0 \\ v_{21} & 1 & 0 \\ v_{31} & 0 & 1 \end{bmatrix},$$

while G_2'' is given by pairs (g, h), where

$$g = \begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that $G_2 = G_2' G_2'' G_{2\mathrm{red}}$. Consider the G_2-invariant subset Σ of W_2' of morphisms of the form

$$\begin{bmatrix} q & \ell & 0 \\ v_{21}q + \ell_1 u_{31} & v_{21}\ell + \ell_1 u_{32} & \ell_1 \\ v_{31}q + \ell_2 u_{31} & v_{31}\ell + \ell_2 u_{32} & \ell_2 \end{bmatrix}.$$

Note that W_2 is the subset of injective morphisms inside $W_2' \setminus \Sigma$, so it is open and G_2-invariant. Moreover, it is a proper subset as, for instance, the morphism represented by the matrix

$$\begin{bmatrix} X^2 - Y^2 & X & 0 \\ XZ^2 & Z^2 & Y \\ YZ^2 & 0 & X \end{bmatrix}$$

is in $W_2' \setminus \Sigma$ but is not injective. Our aim is to construct a geometric quotient of $W_2' \setminus \Sigma$ modulo G_2; it will follow that W_2/G_2 exists and is a proper open subset of $(W_2' \setminus \Sigma)/G_2$.

Firstly, we construct the geometric quotient W_2'/G_2'. Because of the conditions on q, ℓ, ℓ_1, ℓ_2, it is easy to see that Σ is a subbundle of W_2' of rank 14. The quotient bundle, denoted E', has rank 18. The quotient map $W_2' \to E'$ is a geometric quotient modulo G_2'. Moreover, the canonical action of $G_2'' G_{2\mathrm{red}}$

on U is E'-linearised and the map $W_2' \to E'$ is $G_2''G_{2\mathrm{red}}$-equivariant. Let σ' be the zero-section of E'. The restricted map $W_2' \setminus \Sigma \to E' \setminus \sigma'$ is also a geometric quotient modulo G_2'.

Secondly, we construct a geometric quotient of E' modulo G_2''. The quotient for the base U can be described explicitly as follows. On V^* we consider the trivial bundle with fibre S^2V^* and the subbundle with fibre vV^* at any point $v \in V^*$. The quotient bundle Q' is the geometric quotient U_1/G_2'' and $U/G_2'' \simeq (U_1/G_2'') \times U_2$. Clearly U is a principal G_2''-bundle over U/G_2''. According to 4.2.14 [6], E' descends to a vector bundle E over U/G_2''. The canonical map $E' \to E$ is a geometric quotient modulo G_2''. The composed map $W_2' \to E' \to E$ is a geometric quotient modulo $G_2'G_2''$. Moreover, the canonical action of $G_{2\mathrm{red}}$ on U/G_2'' is linearised with respect to E and the map $W_2' \to E$ is $G_{2\mathrm{red}}$-equivariant. Let σ be the zero-section of E. The restricted map $W_2' \setminus \Sigma \to E' \setminus \sigma' \to E \setminus \sigma$ is also a geometric quotient modulo $G_2'G_2''$.

Let $x \in U/G_2''$ be a point and let $\xi \in E_x$ be a non-zero vector lying over x. The stabiliser of x in $G_{2\mathrm{red}}$ is S and $S\xi = \mathbb{C}^*\xi$. Thus the canonical map $E \setminus \sigma \to \mathbb{P}(E)$ is a geometric quotient modulo S. It remains to construct a geometric quotient of $\mathbb{P}(E)$ modulo the induced action of $G_{2\mathrm{red}}/S$. Clearly, $(U/G_2'')/(G_{2\mathrm{red}}/S)$ exists and is isomorphic to $\mathbb{P}(Q) \times \mathbb{P}^2$, where Q is the bundle on $\mathbb{P}(V^*)$ to which Q' descends. As noted in the proof of 2.2.5, $\mathbb{P}(Q)$ is the Hilbert scheme of zero-dimensional subschemes of $\mathbb{P}^2$ of length 2. It remains to show that $\mathbb{P}(E)$ descends to a fibre bundle on $\mathbb{P}(Q) \times \mathbb{P}^2$. We consider the character χ of $G_{2\mathrm{red}}$ given by $\chi(g, h) = \det(g)\det(h)^{-1}$. Note that χ is well-defined because it is trivial on homotheties. We multiply the action of $G_{2\mathrm{red}}$ on E by χ and we denote the resulting linearised bundle by E_χ. The action of S on E_χ is trivial, hence E_χ is $G_{2\mathrm{red}}/S$-linearised. The isotropy subgroup in $G_{2\mathrm{red}}/S$ for any point in U/G_2'' is trivial, so we can apply [6], lemma 4.2.15, to deduce that E_χ descends to a vector bundle F over $\mathbb{P}(Q) \times \mathbb{P}^2$. The induced map $\mathbb{P}(E) \to \mathbb{P}(F)$ is a geometric quotient map modulo $G_{2\mathrm{red}}/S$. We conclude that the composed map

$$W_2' \setminus \Sigma \longrightarrow E' \setminus \sigma' \longrightarrow E \setminus \sigma \longrightarrow \mathbb{P}(E) \longrightarrow \mathbb{P}(F)$$

is a geometric quotient map modulo G_2 and that a geometric quotient W_2/G_2 exists and is a proper open subset inside $\mathbb{P}(F)$. $\square$

Proposition 3.2.4. *The geometric quotient W_2/G_2 is isomorphic to X_2.*

Proof. We must construct resolution 3.1.2(ii) starting from the Beilinson spectral sequence for $\mathcal{F}$. We prefer to work, instead, with the sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$,

which gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,4)$. Diagram (2.2.3) [4] for $\mathcal{G}$ takes the form

$$\begin{array}{ccccc} 2\mathcal{O}(-2) & \xrightarrow{\varphi_1} & \mathcal{O}(-1) & & 0 \\ \\ \mathcal{O}(-2) & \xrightarrow{\varphi_3} & 4\mathcal{O}(-1) & \xrightarrow{\varphi_4} & 4\mathcal{O} \end{array} .$$

Since $\mathcal{G}$ maps surjectively onto $\mathcal{C}oker(\varphi_1)$ and is semi-stable, φ_1 cannot be zero and $\mathcal{C}oker(\varphi_1)$ cannot be isomorphic to $\mathcal{O}_L(-1)$ for a line $L \subset \mathbb{P}^2$. Thus $\mathcal{C}oker(\varphi_1)$ is the structure sheaf of a point $x \in \mathbb{P}^2$ and $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$. The exact sequence (2.2.5) [4] reads:

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

We see from this that $\mathcal{C}oker(\varphi_4)$ has no zero-dimensional torsion. The exact sequence (2.2.4) [4] takes the form

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\varphi_3} 4\mathcal{O}(-1) \xrightarrow{\varphi_4} 4\mathcal{O} \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow 0.$$

We claim that φ_3 is equivalent to the morphism represented by the matrix

$$\left[\begin{array}{cccc} X & Y & Z & 0 \end{array}\right]^{\mathrm{T}}.$$

The argument uses the fact that $\mathcal{C}oker(\varphi_4)$ has no zero-dimensional torsion and is analogous to the proof that the vector space H from 2.1.4 has dimension 3. Now we can describe φ_4. We claim that φ_4 is equivalent to a morphism represented by a matrix of the form

$$\left[\begin{array}{rrrc} -Y & X & 0 & \star \\ -Z & 0 & X & \star \\ 0 & -Z & Y & \star \\ 0 & 0 & 0 & \ell \end{array}\right]$$

with $\ell \in V^*$. The argument, we recall from the proof of 3.1.3, uses the fact that the map $4\mathcal{O} \to \mathcal{C}oker(\varphi_4)$ is injective on global sections and the fact that the only morphism $\mathcal{O}_L(1) \to \mathcal{C}oker(\varphi_4)$ for any line $L \subset \mathbb{P}^2$ is the zero-morphism. Indeed, such a morphism must factor through φ_5 because the composed map $\mathcal{O}_L(1) \to \mathcal{C}oker(\varphi_4) \to \mathcal{G}$ is zero. This follows from the fact that both $\mathcal{O}_L(1)$ and $\mathcal{G}$ are semi-stable and $p(\mathcal{O}_L(1)) > p(\mathcal{G})$.

If $\ell = 0$, then $\mathcal{C}oker(\varphi_4)$ would have a direct summand with Hilbert polynomial $P(t) = 2t + 3$. Such a sheaf must map injectively to $\mathcal{G}$, because its intersection with $\mathcal{O}(-3)$ could only be the zero-sheaf. This contradicts the semi-stability of $\mathcal{G}$. Thus $\ell \neq 0$. Let L be the line with equation $\ell = 0$. We obtain the extension

$$0 \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{O}_L \longrightarrow 0,$$

which yields the resolution

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow 0.$$

Write $\mathcal{C} = \mathcal{C}oker(\varphi_5)$. Since $\mathrm{Ext}^1(\mathcal{O}(-3), \mathcal{O}(-1)) = 0$, the morphism φ_5 lifts to a morphism $\mathcal{O}(-3) \to \mathcal{O} \oplus \mathcal{O}(1)$. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{C} \longrightarrow 0.$$

We now apply the horseshoe lemma to the extension $\mathcal{G}$ of $\mathbb{C}_x$ by $\mathcal{C}$, to the above resolution of $\mathcal{C}$ and to the standard resolution of $\mathbb{C}_x$ tensored with $\mathcal{O}(-1)$. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{G} \longrightarrow 0.$$

The morphism $\mathcal{O}(-3) \to \mathcal{O}(-3)$ is non-zero because $\mathrm{H}^1(\mathcal{G})$ vanishes. We may cancel $\mathcal{O}(-3)$ we get the dual of resolution 3.1.2(ii). $\square$

Proposition 3.2.5. *There exists a geometric quotient W_3/G_3, which is isomorphic to the universal quintic inside $\mathbb{P}^2 \times \mathbb{P}(S^5V^*)$. Moreover, W_3/G_3 is isomorphic to X_3, so this is a smooth closed subvariety of $\mathrm{M}_{\mathbb{P}^2}(5, 1)$.*

Proof. For the first part of the proposition we refer to 3.2 [4]. Succinctly, the map of W_3 to the universal quintic given by

$$\left[\begin{array}{cc} \ell_1 & \ell_2 \\ f_1 & f_2 \end{array}\right] \longrightarrow (x, \langle \ell_1 f_2 - \ell_2 f_1 \rangle), \quad \text{where } x \text{ is the zero-set of } \ell_1 \text{ and } \ell_2,$$

is a geometric quotient map. Clearly, the natural morphism $W_3/G_3 \to X_3$ is bijective. In order to show that it is an isomorphism we need to derive resolution 3.1.5 starting from the Beilinson spectral sequence of $\mathcal{F}$ and performing algebraic operations (compare 3.1.6 [4]). By duality, we may also start with the Beilinson spectral sequence for the sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$. Table (2.2.3) [4] for $\mathrm{E}^1(\mathcal{G})$ takes the form

$$3\mathcal{O}(-2) \xrightarrow{\varphi_1} 3\mathcal{O}(-1) \xrightarrow{\varphi_2} \mathcal{O} \ .$$

$$2\mathcal{O}(-2) \xrightarrow{\varphi_3} 6\mathcal{O}(-1) \xrightarrow{\varphi_4} 5\mathcal{O}$$

As in the proof of 2.2.4, we have $\mathcal{K}er(\varphi_2) = \mathcal{I}m(\varphi_1)$ and $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$. The exact sequence (2.2.5) [4]

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{G} \longrightarrow 0$$

yields the resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\eta} \mathcal{O}(-3) \oplus 6\mathcal{O}(-1) \longrightarrow 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

$$\eta = \left[\begin{array}{c} 0 \\ \varphi_3 \end{array}\right].$$

As in the proof of 3.1.3, we can show that any matrix equivalent to the matrix representing φ_3 has three linearly independent entries on each column.

It follows that, modulo elementary operations on rows and columns, φ_3 is represented by a matrix having one of the following forms:

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ X & R \\ Y & S \\ Z & T \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ X & 0 \\ Y & R \\ Z & S \\ 0 & T \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cc} 0 & 0 \\ X & 0 \\ Y & 0 \\ Z & R \\ 0 & S \\ 0 & T \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cc} X & 0 \\ Y & 0 \\ Z & 0 \\ 0 & X \\ 0 & Y \\ 0 & Z \end{array}\right].$$

Here R, S, T form a basis of V^*. As in the proof of 3.1.3, it can be shown that the first three matrices are unfeasible. We are left with the last possibility.

By virtue of [10], lemma 3, taking duals of the locally free sheaves occuring in the above resolution of $\mathcal{G}$ yields a monad with middle cohomology $\mathcal{F}$ of the form

$$0 \longrightarrow 5\mathcal{O}(-2) \longrightarrow 6\mathcal{O}(-1) \oplus \mathcal{O}(1) \xrightarrow{\eta^{\mathrm{T}}} 2\mathcal{O} \longrightarrow 0.$$

From this we get the resolution

$$0 \longrightarrow 5\mathcal{O}(-2) \longrightarrow 2\Omega^1 \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Combining with the standard resolution of Ω^1 yields the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus 5\mathcal{O}(-2) \xrightarrow{\varphi} 6\mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

From the semi-stability of $\mathcal{F}$ we see that $\operatorname{rank}(\varphi_{12}) = 5$, so we may cancel $5\mathcal{O}(-2)$ to get the desired resolution for $\mathcal{F}$. $\square$

3.3. **Geometric description of the strata.** Let $\mathcal{F} = \mathcal{C}oker(\varphi)$ be a sheaf in X_0 with φ as in 3.1.1. We recall that φ_{11} is semi-stable as a Kronecker V-module. We shall decompose X_0 into locally closed subsets according to the kernel of φ_{11}. We have an exact sequence

$$0 \longrightarrow \mathcal{O}(-d) \xrightarrow{\eta} 4\mathcal{O}(-2) \xrightarrow{\varphi_{11}} 3\mathcal{O}(-1) \longrightarrow \mathcal{C}oker(\varphi_{11}) \longrightarrow 0,$$

$$\eta = \left[\begin{array}{cccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{array}\right]^{\mathrm{T}}, \qquad \eta_i = (-1)^i \varphi_i / g,$$

where φ_i is the maximal minor of φ_{11} obtained by deleting the i-th column and $g = \gcd(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$. The maximal minors of a generic morphism φ_{11} have no common factor, i.e. $\mathcal{K}er(\varphi_{11}) \simeq \mathcal{O}(-5)$. We denote by X_{01} and X_{02} the subsets of X_0 for which $\mathcal{K}er(\varphi_{11})$ is isomorphic to $\mathcal{O}(-4)$, respectively to $\mathcal{O}(-3)$. The case $\deg(g) = 3$ is not feasible, because in this case φ_{11} is equivalent to a morphism represented by a matrix with a zero-column, contrary to semi-stability. As before, the superscript $^{\mathrm{D}}$ applied to a subset of $\mathrm{M}_{\mathbb{P}^2}(5,1)$ will signify the corresponding subset of $\mathrm{M}_{\mathbb{P}^2}(5,4)$ obtained by duality.

Proposition 3.3.1. *The sheaves $\mathcal{G}$ from $(X_0 \setminus (X_{01} \cup X_{02}))^{\mathrm{D}} \subset \mathrm{M}_{\mathbb{P}^2}(5,4)$ have the form $\mathcal{J}_Z(3)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length* 6

not contained in a conic curve, contained in a quintic curve C, and $\mathcal{J}_Z \subset \mathcal{O}_C$ is its ideal sheaf.

The generic sheaves $\mathcal{G}$ in X_0^{D} have the form $\mathcal{O}_C(3)(-P_1 - \ldots - P_6)$, where $C \subset \mathbb{P}^2$ is a smooth quintic curve and P_i, $1 \le i \le 6$, are distinct points on C not contained in a conic curve. By duality, the generic sheaves $\mathcal{F}$ in X_0 have the form $\mathcal{O}_C(P_1 + \ldots + P_6)$.

Proof. The sheaves $\mathcal{G}$ from $(X_0 \setminus (X_{01} \cup X_{02}))^{\mathrm{D}}$ are precisely the sheaves with resolution

$$0 \longrightarrow \mathcal{O}(-2) \oplus 3\mathcal{O}(-1) \xrightarrow{\psi} 4\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where ψ_{12} is semi-stable as a Kronecker V-module and its maximal minors have no common factor. According to 4.5 and 4.6 [2], $\mathcal{C}oker(\psi_{12}) \simeq \mathcal{I}_Z(3)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 6 not contained in a conic curve. Conversely, any $\mathcal{I}_Z(3)$ is the cokernel of some ψ_{12} with the above properties. The conclusion now follows as at 2.3.4(i). $\square$

Proposition 3.3.2. *The sheaves $\mathcal{F}$ giving points in X_{02} are precisely the extension sheaves*

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

satisfying $\mathrm{H}^1(\mathcal{F}) = 0$. Here C' and C are arbitrary cubic, respectively conic curves in $\mathbb{P}^2$.

Proof. Assume that $\mathcal{F}$ is in X_{02}, i.e. $\mathcal{K}er(\varphi_{11}) \simeq \mathcal{O}(-3)$. The entries of η span V^*, otherwise the semi-stability of φ_{11}, as a Kronecker V-module, would get contradicted. For instance, if

$$\eta \sim \left[\begin{array}{c} X \\ Y \\ 0 \\ 0 \end{array}\right], \qquad \text{then} \qquad \varphi_{11} \sim \left[\begin{array}{cccc} -Y & X & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{array}\right].$$

Thus $\eta \sim \left[\begin{array}{c} X \\ Y \\ Z \\ 0 \end{array}\right]$, forcing $\varphi_{11} \sim \left[\begin{array}{cccc} -Y & X & 0 & \star \\ -Z & 0 & X & \star \\ 0 & -Z & Y & \star \end{array}\right] = \left[\begin{array}{cc} & \star \\ \rho & \star \\ & \star \end{array}\right]$.

We have an exact sequence

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{C}oker(\rho) \longrightarrow \mathcal{C}oker(\varphi_{11}) \longrightarrow 0$$

hence, since $\mathcal{C}oker(\rho) \simeq \mathcal{O}$, we have an isomorphism $\mathcal{C}oker(\varphi_{11}) \simeq \mathcal{O}_C$ for a conic curve $C \subset \mathbb{P}^2$. Applying the snake lemma to the exact diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & & \mathcal{O} \arrow[d] & & \\
 & 0 \arrow[r] & 4\mathcal{O}(-2) \arrow[r, "\varphi"] \arrow[d, equal] & 3\mathcal{O}(-1) \oplus \mathcal{O} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(-3) \arrow[r] & 4\mathcal{O}(-2) \arrow[r, "\varphi_{11}"] & 3\mathcal{O}(-1) \arrow[r] \arrow[d] & \mathcal{O}_C \arrow[r] & 0 \\
 & & & 0 & &
\end{tikzcd}$$

we get an extension as in the proposition. Conversely, assume we are given an extension

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

satisfying $\mathrm{H}^1(\mathcal{F}) = 0$. We shall first show that there is a resolution for $\mathcal{O}_C$ as in the diagram above. Combining the exact sequences

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow 3\mathcal{O}(-2) \xrightarrow{\rho} 3\mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

we obtain the resolution

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\eta} 4\mathcal{O}(-2) \xrightarrow{\psi} 3\mathcal{O}(-1) \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

We need to prove that ψ is semi-stable as a Kronecker V-module. Since η has three linearly independent entries, ψ must have three linearly independent maximal minors, and this rules out the cases when ψ could be equivalent to a matrix having a zero-column or a zero-submatrix of size 2×2. It remains to rule out the case

$$\psi = \begin{bmatrix} -Y & X & 0 & R \\ -Z & 0 & X & S \\ 0 & 0 & 0 & T \end{bmatrix}. \qquad \text{Denote} \quad \xi = \begin{bmatrix} -Y & X & 0 \\ -Z & 0 & X \end{bmatrix}$$

and let L_1, L_2 be the lines with equations $X = 0$, respectively $T = 0$. The snake lemma applied to the exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}(-3) \arrow[r] \arrow[d, equal] & 3\mathcal{O}(-2) \arrow[r, "\xi"] \arrow[d] & 2\mathcal{O}(-1) \arrow[r] \arrow[d] & \mathcal{O}_{L_1} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(-3) \arrow[r] & 4\mathcal{O}(-2) \arrow[r, "\psi"] \arrow[d] & 3\mathcal{O}(-1) \arrow[r] \arrow[d] & \mathcal{O}_C \arrow[r] & 0 \\
 & 0 \arrow[r] & \mathcal{O}(-2) \arrow[r, "T"] \arrow[d] & \mathcal{O}(-1) \arrow[r] \arrow[d] & \mathcal{O}_{L_2}(-1) \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

yields an extension

$$0 \longrightarrow \mathcal{O}_{L_1} \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{L_2}(-1) \longrightarrow 0.$$

This gives $h^0(\mathcal{O}_C \otimes \Omega^1(1)) = 1$, which is absurd, namely $\mathrm{H}^0(\mathcal{O}_C \otimes \Omega^1(1))$ vanishes. Thus ψ is semi-stable. We now apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

to the standard resolution of $\mathcal{O}_{C'}$ and to the resolution of $\mathcal{O}_C$ from above. We obtain the exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus 4\mathcal{O}(-2) \longrightarrow 3\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

By hypothesis $\mathrm{H}^1(\mathcal{F})$ vanishes, hence the map $\mathcal{O}(-3) \to \mathcal{O}(-3)$ is non-zero. Cancelling $\mathcal{O}(-3)$ we obtain a resolution as in 3.1.1 in which $\varphi_{11} = \psi$ is a semi-stable Kronecker V-module. We conclude that $\mathcal{F}$ gives a point in X_{02}. □

Let $X_{10} \subset X_1$ be the open subset given by the condition that φ_{12} and φ_{22} have no common linear term. We denote by $X_{11} = X_1 \setminus X_{10}$ the complement.

Proposition 3.3.3. (i) *The sheaves $\mathcal{F}$ giving points in X_{10} are precisely the sheaves $\mathcal{J}_Z(2)$, where $Z \subset \mathbb{P}^2$ is the intersection of two conic curves without common component, Z is contained in a quintic curve $C \subset \mathbb{P}^2$ and $\mathcal{J}_Z \subset \mathcal{O}_C$ is its ideal sheaf.*

The generic sheaves in X_1 are of the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3 - P_4)$, where $C \subset \mathbb{P}^2$ is a smooth quintic curve and P_i, $1 \leq i \leq 4$, are distinct points on C in general linear position.

(ii) *The sheaves $\mathcal{F}$ giving points in X_{11} are precisely the extension sheaves*

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_x(1) \longrightarrow 0$$

satisfying $\mathrm{H}^0(\mathcal{F} \otimes \Omega^1(1)) = 0$. *Here $L \subset \mathbb{P}^2$ is a line and $\mathcal{J}_x \subset \mathcal{O}_C$ is the ideal sheaf of a point x on a quartic curve $C \subset \mathbb{P}^2$.*

Proof. (i) Adopting the notations of 3.1.2(i), we notice that the restriction of φ to $\mathcal{O}(-2)$ has cokernel $\mathcal{I}_Z(2)$, where Z is the subscheme of length 4 in $\mathbb{P}^2$ given by the equations $\varphi_{12} = 0$, $\varphi_{22} = 0$. The sheaves in X_{10} are precisely the cokernels of injective morphisms $\mathcal{O}(-3) \to \mathcal{I}_Z(2)$. Let C be the quintic curve defined by the inclusion $\mathcal{O}(-3) \subset \mathcal{I}_Z(2) \subset \mathcal{O}(2)$. We have $\mathcal{F} \simeq \mathcal{J}_Z(2)$.

(ii) Let us write $\varphi_{12} = \ell\psi_{12}$, $\varphi_{22} = \ell\psi_{22}$, with $\ell, \psi_{12}, \psi_{22}$ non-zero one-forms, ψ_{12} and ψ_{22} linearly independent. Consider the morphism

$$\psi \colon \mathcal{O}(-3) \oplus \mathcal{O}(-1) \longrightarrow 2\mathcal{O}, \qquad \psi = \left[\begin{array}{cc} \varphi_{11} & \psi_{12} \\ \varphi_{21} & \psi_{22} \end{array} \right].$$

$\mathcal{C}oker(\psi)$ is isomorphic to a sheaf of the form $\mathcal{J}_x(1)$ as in the proposition. Conversely, any sheaf $\mathcal{J}_x(1)$ is the cokernel of some injective morphism ψ with linearly independent entries ψ_{12} and ψ_{22}. Let L be the line with equation $\ell = 0$. We apply the snake lemma to the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(-3) \oplus \mathcal{O}(-2) \arrow[r, "\varphi"] \arrow[d, "\alpha"] & 2\mathcal{O} \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] & 0, \\ 0 \arrow[r] & \mathcal{O}(-3) \oplus \mathcal{O}(-1) \arrow[r, "\psi"] & 2\mathcal{O} \arrow[r] & \mathcal{J}_x(1) \arrow[r] & 0 \end{tikzcd}$$

$$\alpha = \left[\begin{array}{cc} 1 & 0 \\ 0 & \ell \end{array} \right].$$

As $\mathcal{C}oker(\alpha) \simeq \mathcal{O}_L(-1)$, we get the extension

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_x(1) \longrightarrow 0.$$

Conversely, assume that $\mathcal{F}$ is an extension of $\mathcal{J}_x(1)$ by $\mathcal{O}_L(-1)$ satisfying the condition $\mathrm{H}^0(\mathcal{F} \otimes \Omega^1(1)) = 0$. Combining the resolutions for these two sheaves we get the exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Our cohomological condition in the hypothesis ensures that $\mathcal{O}(-1)$ may be cancelled, hence we obtain a resolution as in 3.1.2(i) with $\varphi_{12} = \ell\psi_{12}$ and $\varphi_{22} = \ell\psi_{22}$. Thus $\mathcal{F}$ gives a point in X_{11}. $\square$

Proposition 3.3.4. *The generic sheaves from X_2 are precisely the non-split extension sheaves*

$$0 \longrightarrow \mathcal{J}_x(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

for which there is a global section of $\mathcal{F}(1)$ taking the value 1 at every point of Z. Here $\mathcal{J}_x \subset \mathcal{O}_C$ is the ideal sheaf of a point x on a quintic curve $C \subset \mathbb{P}^2$ and $Z \subset C$ is the union of two distinct points, also distinct from x.

There is an open subset of X_2 consisting of the isomorphism classes of all sheaves of the form $\mathcal{O}_C(1)(-P_1+P_2+P_3)$, where $C \subset \mathbb{P}^2$ is a smooth quintic curve and P_1, P_2, P_3 are distinct points on C. In particular, X_2 lies in the closure of X_1 and X_3 lies in the closure of X_2.

Proof. One direction was proven at 3.1.2(ii). Given $\mathcal{F}$ in X_2, there is an extension as in the proposition with x given by the equations $\ell_1 = 0$, $\ell_2 = 0$, Z given by the equations $q = 0$, $\ell = 0$ and C given by the equation $\det(\varphi) = 0$.

For the converse we apply the horseshoe lemma to the resolutions

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{I}_x(1) \longrightarrow \mathcal{J}_x(1) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}(-4) \xrightarrow{\zeta} \mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\xi} \mathcal{O}(-1) \xrightarrow{\pi} \mathcal{O}_Z \longrightarrow 0,$$

$$\zeta = \left[\begin{array}{c} -\ell \\ q \end{array}\right], \qquad \xi = \left[\begin{array}{cc} q & \ell \end{array}\right].$$

By hypothesis, π lifts to a morphism $\alpha\colon \mathcal{O}(-1) \to \mathcal{F}$. We define morphisms β, γ, δ as at 2.3.2. By the reason given there, δ is non-zero, namely, if δ were zero, then the extension for $\mathcal{F}$ would split. We arrive at the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{I}_x(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

which, further, yields resolution 3.1.2(ii).

Assume now that C is smooth and write $x = P_1$, $Z = \{P_2, P_3\}$. The only non-trivial extension sheaf of $\mathcal{O}_Z$ by $\mathcal{J}_x(1)$ is isomorphic to the sheaf $\mathcal{F} = \mathcal{O}_C(1)(-P_1+P_2+P_3)$. We must show that $\mathcal{F}(1)$ has a global section that does not vanish at P_2 and P_3. We argue as at 2.3.2. Let $\varepsilon_2, \varepsilon_3\colon \mathrm{H}^0(\mathcal{O}_Z) \to \mathbb{C}$ be the be the linear forms of evaluation at P_2, P_3. Let $\delta\colon \mathrm{H}^0(\mathcal{O}_Z) \to \mathrm{H}^1(\mathcal{J}_x(2))$ be the connecting homomorphism in the long exact cohomology sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(2)(-x) \longrightarrow \mathcal{F}(1) \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We must show that neither ε_2 nor ε_3 is orthogonal to $\mathrm{Ker}(\delta)$. This is equivalent to saying that neither ε_2 nor ε_3 are in the image of the dual map δ^*. By Serre

duality δ^* is the restriction morphism

$$\begin{array}{ccc}
\mathrm{H}^0(\mathcal{O}_C(-2)(x)\otimes\omega_C) & \longrightarrow & \mathrm{H}^0((\mathcal{O}_C(-2)(x)\otimes\omega_C)|_Z)\ . \\
\| & & \| \\
\mathrm{H}^0(\mathcal{O}_C(x)) & & \mathrm{H}^0(\mathcal{O}_C(x)|_Z) \\
\| & & \| \\
\mathrm{H}^0(\mathcal{O}_C)\simeq\mathbb{C} & \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} & \mathbb{C}^2\simeq\mathrm{H}^0(\mathcal{O}_C|_Z)
\end{array}$$

The linear forms ε_2 and ε_3 correspond to the vectors $(1,0)$ and $(0,1)$ in $\mathbb{C}^2$, so they are clearly not in the image of δ^*. The identity $\mathrm{H}^0(\mathcal{O}_C(x))=\mathrm{H}^0(\mathcal{O}_C)$ follows from the fact that there is no rational function on C that has exactly one pole of multiplicity 1. If this were the case, C would have genus 0.

To see that $X_2\subset\overline{X}_1$ choose a point in X_2 represented by the sheaf $\mathcal{O}_C(1)(-P_1+P_2+P_3)$. We may assume that P_1,P_2,P_3 are non-colinear and that the line through P_2 and P_3 intersects C at five distinct points denoted P_2,P_3,Q_1,Q_2,Q_3. Then $\mathcal{O}_C(1)(-P_1+P_2+P_3)$ is isomorphic to $\mathcal{O}_C(2)(-P_1-Q_1-Q_2-Q_3)$. Clearly, we can find points R_1,R_2,R_3 on C, converging to Q_1,Q_2,Q_3 respectively, such that P_1,R_1,R_2,R_3 are in general linear position. Thus $\mathcal{O}_C(2)(-P_1-R_1-R_2-R_3)$ gives a point in X_1 converging to the chosen point in X_2.

According to 3.1.5, the generic sheaves in X_3 have the form $\mathcal{O}_C(1)(P)$, where $C\subset\mathbb{P}^2$ is a smooth quintic curve and P is a point on C. Choose distinct points P_1,P_2 on C, which are also distinct from P, such that P_2 converges to P_1. The stable-equivalence class of $\mathcal{O}_C(1)(-P_1+P_2+P)$ is in X_2 and converges to the stable-equivalence class of $\mathcal{O}_C(1)(P)$. We conclude that $X_3\subset\overline{X}_2$. □

The following result will be helpful in the discussion about sheaves from X_{01}, which we have left for the end.

Proposition 3.3.5. *Let $\psi\colon 4\mathcal{O}(-2)\to 3\mathcal{O}(-1)$ be a Kronecker V-module. Let ψ_i, $1\le i\le 4$, denote the maximal minor of ψ obtained by deleting the i-th column. Assume that the minors ψ_i have a common linear factor. Then $\mathcal{K}er(\psi)\simeq\mathcal{O}(-4)$ and ψ is semi-stable if and only if ψ_i, $1\le i\le 4$, are linearly independent three-forms.*

Proof. Assume that $\mathcal{K}er(\psi)\simeq\mathcal{O}(-4)$ and that ψ is semi-stable. We argue by contradiction. If the maximal minors of ψ were linearly dependent, then, performing possibly column operations on ψ, we could assume that one of them is zero, say $\psi_4=0$. Let ψ' be the matrix obtained from ψ by deleting the fourth column. It is easy to see that ψ' is semi-stable as a Kronecker V-module. It follows that ψ' is equivalent to the morphism represented by

the matrix

$$\left[\begin{array}{rrr} -Y & X & 0 \\ -Z & 0 & X \\ 0 & -Z & Y \end{array}\right]. \qquad \text{Thus the vector} \qquad \left[\begin{array}{c} X \\ Y \\ Z \\ 0 \end{array}\right]$$

is in the kernel of ψ. This contradicts our hypothesis that $\mathcal{K}er(\psi)$ be isomorphic to $\mathcal{O}(-4)$.

Conversely, assume that ψ_i, $1 \le i \le 4$, are linearly independent. Then they cannot have a common factor of degree 2, that is, in view of the comments at the beginning of this subsection, we have $\mathcal{K}er(\psi) \simeq \mathcal{O}(-4)$. The semi-stability of ψ is also clear: if ψ were equivalent to a matrix having a zero-column, then the ψ_i would span a vector space of dimension at most 1. If ψ were equivalent to a matrix having a zero-submatrix of size 2×2, then the ψ_i would span a vector space of dimension at most two. If ψ were equivalent to a matrix having a zero-submatrix of size 1×3, then the ψ_i would span a vector space of dimension at most 3. □

Proposition 3.3.6. *The sheaves $\mathcal{F}$ giving points in X_{01} occur as non-split extension sheaves of one of the following three kinds:*

(i)
$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L \longrightarrow 0,$$

where $\mathrm{H}^1(\mathcal{F}) = 0$. Here $L \subset \mathbb{P}^2$ is a line and $\mathcal{G}$ is in the exceptional divisor of $\mathrm{M}_{\mathbb{P}^2}(4,0)$. For fixed L and $\mathcal{G}$ the feasible extension sheaves form a locally closed subset of $\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_L, \mathcal{G}))$.

(ii)
$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

Here $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 3 not contained in a line and $\mathcal{E}$ is a sheaf in $\mathrm{M}_{\mathbb{P}^2}(5,-2)$ such that $\mathcal{E}(1)$ belongs to the stratum X_3 of $\mathrm{M}_{\mathbb{P}^2}(5,3)$.

(iii)
$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_Z(1)^{\mathrm{D}} \longrightarrow 0.$$

Here $L \subset \mathbb{P}^2$ is a line and $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 3 not contained in a line, contained in a quartic curve $C \subset \mathbb{P}^2$, and $\mathcal{J}_Z \subset \mathcal{O}_C$ is its ideal sheaf. For fixed $\mathcal{J}_Z$ and L the feasible extension sheaves form a locally closed subset of $\mathbb{P}(\mathrm{Ext}^1(\mathcal{J}_Z(1)^{\mathrm{D}}, \mathcal{O}_L(1)))$.

Proof. Let $\mathcal{F}$ give a point in X_{01}. Recall resolution 3.1.1. We have the isomorphism $\mathcal{K}er(\varphi_{11}) \simeq \mathcal{O}(-4)$ and we denote $\mathcal{C} = \mathcal{C}oker(\varphi_{11})$. We have $P_{\mathcal{C}}(t) = t + 3$, so this sheaf is the direct sum of a zero-dimensional sheaf and $\mathcal{O}_L(d)$ for a line $L \subset \mathbb{P}^2$ and an integer d. It is thus clear that $\mathcal{C}$ has a subsheaf $\mathcal{C}'$ with Hilbert polynomial $P_{\mathcal{C}'}(t) = t + 2$.

Applying the snake lemma to a diagram similar to the first diagram in the proof of 3.3.2 we obtain an extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow \mathcal{C} \longrightarrow 0,$$

where $C \subset \mathbb{P}^2$ is a quartic curve. Let $\mathcal{F}' \subset \mathcal{F}$ be the preimage of $\mathcal{C}'$. We have $P_{\mathcal{F}'}(t) = 5t$ and it is easy to see that $\mathcal{F}'$ is semi-stable. We now use the possible resolutions for sheaves in $\mathrm{M}_{\mathbb{P}^2}(5,0)$ found in section 4, which we obtain independently of any result in this subsection. Taking into account that $\mathrm{H}^0(\mathcal{F}' \otimes \Omega^1(1)) = 0$ leaves only two possible resolutions, the ones at 4.1.2 and 4.1.3. The first resolution must fit into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 5\mathcal{O}(-2) \arrow[r, "\psi"] \arrow[d, "\beta"] & 5\mathcal{O}(-1) \arrow[r] \arrow[d, "\alpha"] & \mathcal{F}' \arrow[r] \arrow[d] & 0\ . \\
0 \arrow[r] & 4\mathcal{O}(-2) \arrow[r, "\varphi"] & 3\mathcal{O}(-1) \oplus \mathcal{O} \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

Since $\alpha(1)$ is injective on global sections, we have one of the following two possibilities:

$$\alpha \sim \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & X & Y & Z \end{array} \right] \qquad \text{or} \qquad \alpha \sim \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & X & Y \end{array} \right].$$

In the first case $\mathcal{K}er(\alpha)$ is isomorphic to Ω^1, so the latter is isomorphic to a direct sum of copies of $\mathcal{O}(-2)$. This is absurd. In the second case we have $\mathcal{K}er(\beta) \simeq \mathcal{O}(-2)$, hence, without loss of generality, we may assume that β is the projection onto the first four terms. From the commutativity of the diagram we get

$$\psi = \left[\begin{array}{ccccc} & & & & 0 \\ & \varphi_{11} & & & 0 \\ & & & & 0 \\ \star & \star & \star & \star & -Y \\ \star & \star & \star & \star & X \end{array} \right].$$

This shows that $\mathcal{F}'$ maps surjectively onto the cokernel of φ_{11}. But this is impossible because, by construction, the image of $\mathcal{F}'$ in $\mathcal{C}$ is the proper subsheaf $\mathcal{C}'$. Thus far we have shown that resolution 4.1.2 for $\mathcal{F}'$ is unfeasible. It remains to examine resolution 4.1.3. This fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \arrow[r, "\psi"] \arrow[d, "\beta"] & 2\mathcal{O}(-1) \oplus \mathcal{O} \arrow[r] \arrow[d, "\alpha"] & \mathcal{F}' \arrow[r] \arrow[d] & 0\ . \\
0 \arrow[r] & 4\mathcal{O}(-2) \arrow[r, "\varphi"] & 3\mathcal{O}(-1) \oplus \mathcal{O} \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

Since α and $\alpha(1)$ are injective on global sections, we see that α and β are injective and we may write

$$\beta = \left[\begin{array}{ccc} -\ell_2 & 0 & 0 \\ \ell_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right], \qquad \alpha = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right].$$

From the commutativity of the diagram and the semi-stability of φ_{11}, we see that ℓ_1 and ℓ_2 are linearly independent one-forms and

$$\varphi_{11} = \left[\begin{array}{cccc} \ell_1 & \ell_2 & 0 & 0 \\ \star & \star & \multicolumn{2}{c}{\multirow{2}{*}{ξ}} \\ \star & \star & & \end{array} \right].$$

We recall that the greatest common divisor of the maximal minors of φ_{11} is a linear form g. Since g divides both $\ell_1 \det(\xi)$ and $\ell_2 \det(\xi)$, we see that g divides $\det(\xi)$, hence ξ is equivalent to a matrix having a zero-entry. Thus we may write

$$\varphi_{11} = \left[\begin{array}{cccc} \ell_1 & \ell_2 & 0 & 0 \\ \star & \star & \xi_3 & 0 \\ \star & \star & \star & \xi_4 \end{array} \right] = \left[\begin{array}{cccc} \multicolumn{3}{c}{\multirow{2}{*}{ζ}} & 0 \\ & & & 0 \\ \star & \star & \star & \xi_4 \end{array} \right].$$

It is clear that ζ is semi-stable as a Kronecker V-module. Assume that the maximal minors of ζ have a common linear factor, say Z. We may then write

$$\varphi = \left[\begin{array}{cccc} X & Z & 0 & 0 \\ Y & 0 & Z & 0 \\ \star & \star & \star & S \\ \star & \star & \star & T \end{array} \right] = \left[\begin{array}{cccc} \multicolumn{3}{c}{\multirow{3}{*}{φ'}} & 0 \\ & & & 0 \\ & & & S \\ \star & \star & \star & T \end{array} \right].$$

Notice that g is a multiple of Z, S is non-zero and does not divide $\det(\varphi')/Z$. We have $\mathcal{C}oker(\zeta) \simeq \mathcal{O}_L$, where $L \subset \mathbb{P}^2$ is the line with equation $Z = 0$. We apply the snake lemma to the exact diagram from below in order to obtain a non-split extension of the form

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L \longrightarrow 0,$$

where $\mathcal{G}$ has resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

with $\psi_{12} = S$ different from zero. From 5.2.1 [4] we see that $\mathcal{G}$ is in the exceptional divisor of $\mathrm{M}_{\mathbb{P}^2}(4,0)$. Conversely, any $\mathcal{G}$ of $\mathrm{M}_{\mathbb{P}^2}(4,0)$, which is in the exceptional divisor, i.e. satisfying the condition $h^0(\mathcal{G}) = 1$, occurs as the cokernel of a morphism ψ as above with $\psi_{12} \neq 0$. Assume now that $\mathcal{F}$ is an extension of $\mathcal{O}_L$ with a sheaf $\mathcal{G}$ as above, satisfying $\mathrm{H}^1(\mathcal{F}) = 0$. Choose an equation $Z = 0$ for L. We combine the resolution of $\mathcal{G}$ with the resolution

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow 3\mathcal{O}(-2) \xrightarrow{\zeta} 2\mathcal{O}(-1) \longrightarrow \mathcal{O}_L \longrightarrow 0$$

Diagram for the snake lemma.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathcal{O}(-2) \arrow[r, "\left[\begin{smallmatrix} S \\ T \end{smallmatrix}\right]"] \arrow[d] & \mathcal{O}(-1) \oplus \mathcal{O} \arrow[d] & & \\
 & 0 \arrow[r] & 4\mathcal{O}(-2) \arrow[r, "\varphi"] \arrow[d] & 3\mathcal{O}(-1) \oplus \mathcal{O} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(-3) \arrow[r] & 3\mathcal{O}(-2) \arrow[r, "\zeta"] \arrow[d] & 2\mathcal{O}(-1) \arrow[r] \arrow[d] & \mathcal{O}_L \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

and we obtain a resolution

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus 4\mathcal{O}(-2) \longrightarrow 3\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

The morphism $\mathcal{O}(-3) \to \mathcal{O}(-3)$ in the above complex is non-zero because, by hypothesis, $\mathrm{H}^1(\mathcal{F})$ vanishes. Thus we may cancel $\mathcal{O}(-3)$ to get resolution 3.1.1 with

$$\varphi = \left[\begin{array}{cccc} & \zeta & & 0 \\ & & & 0 \\ \star & \star & \star & \psi_{12} \\ \star & \star & \star & \psi_{22} \end{array} \right] = \left[\begin{array}{cccc} & \varphi' & & 0 \\ & & & 0 \\ & & & \psi_{12} \\ \star & \star & \star & \psi_{22} \end{array} \right].$$

In view of 3.3.5, the condition that $\mathcal{F}$ be in X_{01} is equivalent to saying that $\det(\varphi')/Z, \psi_{12}X, \psi_{12}Y, \psi_{12}Z$ are linearly independent two-forms. This defines an open subset inside the closed set of extension sheaves of $\mathcal{O}_L$ by $\mathcal{G}$ with vanishing first cohomology.

It remains to examine the case when the maximal minors of ζ have no common factor. Then g is a multiple of ξ_4. We have $\mathcal{K}er(\zeta) \simeq \mathcal{O}(-4)$. According to 4.5 and 4.6 [2], the cokernel of ζ is isomorphic to the structure sheaf of a zero-dimensional scheme Z of length 3 not contained in a line. Write as above

$$\varphi = \left[\begin{array}{cccc} & \zeta & & 0 \\ & & & 0 \\ \star & \star & \star & S \\ \star & \star & \star & T \end{array} \right]$$

and note that the snake lemma gives an extension

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

where $\mathcal{E}$ has a resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0$$

in which $\psi_{12} = S$ and $\psi_{22} = T$. We have $P_{\mathcal{E}}(t) = 5t - 2$. According to 2.1.4, $\mathcal{E}$ is in $\mathrm{M}_{\mathbb{P}^2}(5, -2)$ precisely if S does not divide T. In that case $\mathcal{E}(1)$ gives a point in the stratum X_3 of $\mathrm{M}_{\mathbb{P}^2}(5, 3)$. Finally, assume that S divides T. We have a non-split extension of sheaves

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow 0,$$

where $L \subset \mathbb{P}^2$ is given by the equation $S = 0$ and $\mathcal{S}$ has a resolution of the form

$$0 \longrightarrow 3\mathcal{O}(-2) \xrightarrow{\psi} 2\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{S} \longrightarrow 0,$$

where $\psi_{11} = \zeta$. According to 3.3.2 [4], the subset of $\mathrm{M}_{\mathbb{P}^2}(4, 3)$ of sheaves of the form $\mathcal{S}^{\mathrm{D}}(1)$ is an open subset consisting of all sheaves of the form $\mathcal{J}_Z(2)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 3 not contained in a line, contained in a quartic curve $C \subset \mathbb{P}^2$ and $\mathcal{J}_Z \subset \mathcal{O}_C$ is its ideal sheaf. Assume we are given $\mathcal{S}$ as above, $L \subset \mathbb{P}^2$ a line with equation $S = 0$ and $\mathcal{F}$ a non-split extension of $\mathcal{S}$ by $\mathcal{O}_L(-1)$. We combine the above resolution for $\mathcal{S}$ with the standard resolution of $\mathcal{O}_L(-1)$ to get a resolution for $\mathcal{F}$ as in 3.1.1. By 3.3.5, the condition that $\mathcal{F}$ be in X_{01} is equivalent to saying that S divides $\det(\varphi')$ and $\det(\varphi')/S$ together with the maximal minors of ζ form a linearly independent set in S^2V^*. These conditions define a locally closed subset of $\mathbb{P}(\mathrm{Ext}^1(\mathcal{S}, \mathcal{O}_L(-1)))$. $\square$

From what was said above we can summarise:

Proposition 3.3.7. $\{X_0, X_1, X_2, X_3\}$ *represents a stratification of* $\mathrm{M}_{\mathbb{P}^2}(5, 1)$ *by locally closed subvarieties of codimensions* $0, 2, 3, 5$.

4. Euler characteristic zero

4.1. **Locally free resolutions for semi-stable sheaves.**

Proposition 4.1.1. *Every sheaf* $\mathcal{F}$ *giving a point in* $\mathrm{M}_{\mathbb{P}^2}(5, 0)$ *and satisfying the condition* $h^0(\mathcal{F}(-1)) > 0$ *is of the form* $\mathcal{O}_C(1)$ *for a quintic curve* $C \subset \mathbb{P}^2$.

Proof. Consider a non-zero morphism $\mathcal{O} \to \mathcal{F}(-1)$. As in the proof of 2.1.3 [4], it must factor through an injective morphism $\mathcal{O}_C \to \mathcal{F}(-1)$. Here $C \subset \mathbb{P}^2$ is a curve; its degree must be 5, otherwise $\mathcal{O}_C$ would destabilise $\mathcal{F}(-1)$. As both $\mathcal{O}_C$ and $\mathcal{F}(-1)$ have the same Hilbert polynomial, the injective morphism from above must be an isomorphism.

The converse follows from the general fact that the structure sheaf of a curve in $\mathbb{P}^2$ is stable. $\square$

Proposition 4.1.2. *The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(5,0)$ and satisfying the condition $h^1(\mathcal{F})=0$ are precisely the sheaves with resolution*

$$0 \longrightarrow 5\mathcal{O}(-2) \xrightarrow{\varphi} 5\mathcal{O}(-1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Moreover, such a sheaf $\mathcal{F}$ is properly semi-stable if and only if φ is equivalent to a morphism of the form

$$\left[\begin{array}{cc} \star & \psi \\ \star & 0 \end{array}\right] \quad \text{for some} \quad \psi\colon m\mathcal{O}(-2) \longrightarrow m\mathcal{O}(-1), \quad 1 \le m \le 4.$$

Proof. Assume that $\mathcal{F}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,0)$ and its first cohomology vanishes. For a suitable line $L \subset \mathbb{P}^2$ we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}(1) \longrightarrow \mathcal{F}(1)|_L \longrightarrow 0.$$

The associated long cohomology sequence shows that $\mathrm{H}^1(\mathcal{F}(1))$ vanishes, too. The same argument applied to the exact sequence

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \otimes V \longrightarrow \mathcal{F} \otimes \Omega^1(2) \longrightarrow 0$$

shows that $\mathrm{H}^1(\mathcal{F} \otimes \Omega^1(2)) = 0$. The Beilinson free monad (2.2.1) [4] for $\mathcal{F}(1)$ gives the resolution

$$0 \longrightarrow 5\mathcal{O}(-1) \longrightarrow 5\mathcal{O} \longrightarrow \mathcal{F}(1) \longrightarrow 0.$$

Conversely, assume that $\mathcal{F}$ is the cokernel of a morphism φ as in the proposition. Trivially, $\mathcal{F}$ has no zero-dimensional torsion, because it has a locally free resolution of length 1. For any subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $\mathrm{H}^0(\mathcal{F}')=0$ because the corresponding cohomology group for $\mathcal{F}$ vanishes. We get $\chi(\mathcal{F}') \le 0$, hence $p(\mathcal{F}') \le 0 = p(\mathcal{F})$ and we conclude that $\mathcal{F}$ is semi-stable.

To finish the proof we must show that for properly semi-stable sheaves $\mathcal{F}$ the morphism φ has the special form given in the proposition. Consider a proper subsheaf $\mathcal{F}' \subset \mathcal{F}$ which gives a point in $\mathrm{M}_{\mathbb{P}^2}(m,0)$, $1 \le m \le 4$. $\mathrm{H}^0(\mathcal{F}')$ vanishes, hence also $\mathrm{H}^1(\mathcal{F}')$ vanishes and, repeating the above steps with $\mathcal{F}'$ instead of $\mathcal{F}$, we arrive at a resolution

$$0 \longrightarrow m\mathcal{O}(-2) \xrightarrow{\psi} m\mathcal{O}(-1) \longrightarrow \mathcal{F}' \longrightarrow 0.$$

This fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & m\mathcal{O}(-2) \arrow[r, "\psi"] \arrow[d, "\beta"] & m\mathcal{O}(-1) \arrow[r] \arrow[d, "\alpha"] & \mathcal{F}' \arrow[r] \arrow[d] & 0 \, . \\
0 \arrow[r] & 5\mathcal{O}(-2) \arrow[r, "\varphi"] & 5\mathcal{O}(-1) \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

Since $\alpha(1)$ is injective on global sections we see that α, hence also β, are injective. Thus φ has the required special form. $\square$

Proposition 4.1.3. *The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(5,0)$ and satisfying the cohomological conditions $h^0(\mathcal{F}(-1)) = 0$ and $h^1(\mathcal{F}) = 1$ are precisely the sheaves with resolution*

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi_{12} \colon 2\mathcal{O}(-2) \to 2\mathcal{O}(-1)$ is an injective morphism.

Proof. The Beilinson free monad (2.2.1) [4] for $\mathcal{F}$ reads as follows:

$$0 \longrightarrow 5\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow (m+5)\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow 0.$$

From this we obtain the exact sequences

$$0 \longrightarrow 5\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow \Omega^1 \oplus (m+2)\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}(-3) \oplus 5\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O}(-2) \oplus (m+2)\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

with $\varphi_{13} = 0$, $\varphi_{23} = 0$. As in the proof of 2.1.4, we see that $\operatorname{rank}(\varphi_{12}) = 3$, so we may cancel $3\mathcal{O}(-2)$ to get the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} (m+2)\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

with $\varphi_{13} = 0$. By the injectivity of φ we must have $m \le 1$. If $m = 1$, then $\mathcal{F}$ has a subsheaf of the form $\mathcal{O}_L$, for a line $L \subset \mathbb{P}^2$, contrary to semi-stability. We conclude that $m = 0$ and we obtain a resolution as in the proposition. If φ_{12} were not injective, then φ_{12} would be equivalent to a morphism represented by a matrix with a zero-row or a zero-column. Thus $\mathcal{F}$ would have a destabilising subsheaf of the form $\mathcal{O}_C$ or a destabilising quotient sheaf of the form $\mathcal{O}_C(-1)$ for a conic curve $C \subset \mathbb{P}^2$.

Conversely, we assume that $\mathcal{F}$ has a resolution as in the proposition and we need to show that there are no destabilising subsheaves $\mathcal{E}$. Such a subsheaf must satisfy $h^0(\mathcal{E}) = 1$, $h^1(\mathcal{E}) = 0$, $P_{\mathcal{E}}(t) = mt + 1$, $1 \le t \le 4$. Moreover, $\mathrm{H}^0(\mathcal{E}(-1))$ and $\mathrm{H}^0(\mathcal{E} \otimes \Omega^1(1))$ vanish because the corresponding cohomology groups for $\mathcal{F}$ vanish. We can now write the Beilinson free monad for $\mathcal{E}$. We get a resolution that fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (m-1)\mathcal{O}(-2) \arrow[r, "\psi"] \arrow[d, "\beta"] & (m-2)\mathcal{O}(-1) \oplus \mathcal{O} \arrow[r] \arrow[d, "\alpha"] & \mathcal{E} \arrow[r] \arrow[d] & 0\,. \\
0 \arrow[r] & \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \arrow[r, "\varphi"] & 2\mathcal{O}(-1) \oplus \mathcal{O} \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

Since α and $\alpha(1)$ are injective on global sections, we see that α is injective, forcing β to be injective, too. Thus $m = 2$ or $m = 3$. In both cases φ_{12} fails to be injective, contradicting our hypothesis. □

Proposition 4.1.4. *The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(5,0)$ and satisfying the cohomological conditions $h^0(\mathcal{F}(-1)) = 0$ and $h^1(\mathcal{F}) = 2$ are precisely the sheaves with resolution*

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

such that φ_{11} has linearly independent entries and, likewise, φ_{22} has linearly independent entries.

Proof. Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(5,0)$ and satisfy the conditions from the proposition. The Beilinson free monad for $\mathcal{F}$ reads

$$0 \longrightarrow 5\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow (m+5)\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow 2\mathcal{O} \longrightarrow 0.$$

Dualising and tensoring with $\mathcal{O}(1)$ we get the following resolution for the sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$, which gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,5)$:

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\eta} 2\mathcal{O}(-2) \oplus (m+5)\mathcal{O}(-1) \xrightarrow{\varphi} m\mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

$$\eta = \begin{bmatrix} 0 \\ \psi \end{bmatrix}.$$

Here $\varphi_{12} = 0$. As $\mathcal{G}$ has rank zero and maps surjectively onto $\mathcal{C} = \mathcal{C}oker(\varphi_{11})$, we see that $m \le 2$. If $m = 2$, then φ_{11} must be injective, otherwise $\mathcal{C}$ will have positive rank. We get $P_{\mathcal{C}}(t) = 2t$, hence $\mathcal{C}$ destabilises $\mathcal{G}$. The case $m = 0$ can be eliminated as in the proof of 3.1.3. Thus $m = 1$. As in the proof of 3.2.5, we may assume that ψ is represented by the matrix

$$\begin{bmatrix} X & Y & Z & 0 & 0 & 0 \\ 0 & 0 & 0 & X & Y & Z \end{bmatrix}^{\mathrm{T}}.$$

From the Beilinson monad for $\mathcal{F}$ we obtain the resolution

$$0 \longrightarrow 5\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow 2\Omega^1 \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

which, combined with the standard resolution for Ω^1, yields the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus 5\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 6\mathcal{O}(-2) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Note that $\mathcal{F}$ maps surjectively onto $\mathcal{C}oker(\varphi_{11}, \varphi_{12})$, so this sheaf is supported on a curve, forcing $\mathrm{rank}(\varphi_{12}) \ge 4$. If $\mathrm{rank}(\varphi_{12}) = 4$, then $\mathcal{C}oker(\varphi_{11}, \varphi_{12})$ would have Hilbert polynomial $P(t) = 2t - 2$, so it would destabilise $\mathcal{F}$. We deduce that $\mathrm{rank}(\varphi_{12}) = 5$, so we may cancel $5\mathcal{O}(-2)$ to get a resolution as in the proposition. If the entries of φ_{11} were linearly dependent, then $\mathcal{F}$ would have a destabilising quotient sheaf of the form $\mathcal{O}_L(-2)$ for a line $L \subset \mathbb{P}^2$. If the entries of φ_{22} were linearly dependent, then $\mathcal{F}$ would have a destabilising subsheaf of the form $\mathcal{O}_L$.

Conversely, we assume that $\mathcal{F}$ has a resolution as in the proposition and we need to show that there is no destabilising subsheaf. Let $\mathcal{F}' \subset \mathcal{F}$ be a non-zero subsheaf of multiplicity at most 4. We shall use the extension

$$0 \longrightarrow \mathcal{J}_x(1) \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_z \longrightarrow 0$$

from 4.3.2. Denote by $\mathcal{C}'$ the image of $\mathcal{F}'$ in $\mathbb{C}_z$ and put $\mathcal{K} = \mathcal{F}' \cap \mathcal{J}_x(1)$. Let $\mathcal{A}$ and $\mathcal{O}_S$ be as in the proof of 3.1.2. Recall that S is a curve of degree $d \leq 4$. We can estimate the slope of $\mathcal{F}'$ as in the proof of loc.cit. and we get

$$p(\mathcal{F}') = -\frac{d}{2} + \frac{h^0(\mathcal{C}') - h^0(\mathcal{A}/\mathcal{K})}{5-d} \leq -\frac{d}{2} + \frac{1}{5-d} < 0 = p(\mathcal{F}).$$

We conclude that $\mathcal{F}$ is semi-stable. □

Let X_i, $i = 0, 1, 2, 3$, be the subset of $\mathrm{M}_{\mathbb{P}^2}(5, 0)$ of stable-equivalence classes of sheaves $\mathcal{F}$ as in 4.1.2, 4.1.3, 4.1.4, respectively 4.1.1.

Proposition 4.1.5. *The subsets X_0, X_1, X_2, X_3 are disjoint. The subset of $\mathrm{M}_{\mathbb{P}^2}(5, 0)$ of stable-equivalence classes of properly semi-stable sheaves is included in $X_0 \cup X_1$.*

Proof. Let $\mathcal{F}$ be a properly semi-stable sheaf in $\mathrm{M}_{\mathbb{P}^2}(5, 0)$. We have an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

with $\mathcal{F}'$ giving a point in $\mathrm{M}_{\mathbb{P}^2}(r, 0)$, $\mathcal{F}''$ giving a point in $\mathrm{M}_{\mathbb{P}^2}(s, 0)$, $r + s = 5$. From the description of $\mathrm{M}_{\mathbb{P}^2}(r, 0)$, $1 \leq r \leq 4$, found in [4], we have the relations

$$h^0(\mathcal{F}') = 0 \quad \text{if } r = 1, 2, \qquad h^0(\mathcal{F}') \leq 1 \quad \text{if } r = 3, 4.$$

In all possible situations we get $h^0(\mathcal{F}) \leq 1$, hence the stable-equivalence class of $\mathcal{F}$ is in $X_0 \cup X_1$. Thus all sheaves in X_2 and X_3 are stable, so X_2 is disjoint from the other X_i and the same is true for X_3. It remains to show that X_0 and X_1 are disjoint. Let $\mathcal{F}$ be a properly semi-stable sheaf as in 4.1.2 and let $\mathcal{G}$ be a sheaf in the same class of stable-equivalence as $\mathcal{F}$. Let $\mathcal{F}'$ be one of the terms of a Jordan-Hölder filtration of $\mathcal{F}$. From the proof of 4.1.2 it transpires that $\mathcal{F}'$ has resolution

$$0 \longrightarrow m\mathcal{O}(-2) \longrightarrow m\mathcal{O}(-1) \longrightarrow \mathcal{F}' \longrightarrow 0,$$

for some integer $1 \leq m \leq 4$. Thus $\mathrm{H}^0(\mathcal{F}') = 0$. Any term of a Jordan-Hölder filtration of $\mathcal{G}$ is also a term of a Jordan-Hölder filtration of $\mathcal{F}$, hence its group of global sections vanishes. We deduce that $\mathrm{H}^0(\mathcal{G}) = 0$. Thus $\mathcal{F}$ cannot give a point in X_1. □

Proposition 4.1.6. *There are no sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(5, 0)$ and satisfying the cohomological conditions $h^0(\mathcal{F}(-1)) = 0$ and $h^1(\mathcal{F}) \geq 3$.*

Proof. In view of 4.1.5, we may restrict our attention to stable sheaves $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(5, 0)$. Suppose that $\mathcal{F}$ satisfies $h^0(\mathcal{F}(-1)) = 0$ and $h^1(\mathcal{F}) \neq 0$. Consider a non-zero morphism $\mathcal{O} \to \mathcal{F}$. As in the proof of 2.1.3 [4], this must factor through an injective morphism $\mathcal{O}_C \to \mathcal{F}$, where $C \subset \mathbb{P}^2$ is a curve. From the stability of $\mathcal{F}$ we see that C can only have degree 4 or 5.

Assume that C has degree 5. The quotient sheaf $\mathcal{C} = \mathcal{F}/\mathcal{O}_C$ is supported on finitely many points and has length 5. Take a subsheaf $\mathcal{C}' \subset \mathcal{C}$ of length 4, and let $\mathcal{F}'$ be its preimage in $\mathcal{F}$. We get an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathbb{C}_x$ is the structure sheaf of a point. Any destabilising subsheaf of $\mathcal{F}'$ would ruin the stability of $\mathcal{F}$, hence $\mathcal{F}'$ is in $\mathrm{M}_{\mathbb{P}^2}(5,-1)$. From subsection 3.1 we know that $h^0(\mathcal{F}') \le 2$, hence $h^0(\mathcal{F}) \le 2$ unless $h^0(\mathcal{F}') = 2$ and the morphism $\mathcal{F} \to \mathbb{C}_x$ is surjective on global sections. In this case we can apply the horseshoe lemma to the above extension, to the standard resolution of $\mathbb{C}_x$ and to the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-1) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

We obtain a resolution

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-4) \oplus 3\mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

which yields an exact sequence

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{C}oker(\varphi_{12}) \longrightarrow \mathcal{F} \longrightarrow 0.$$

We claim that the morphism $\mathcal{O}(-2) \to 3\mathcal{O}(-1)$ in the above resolution is equivalent to the morphism represented by the matrix

$$\left[\begin{array}{c} X \\ Y \\ Z \end{array}\right].$$

The argument uses the fact that $\mathcal{F}$ has no zero-dimensional torsion and is analogous to the proof that the vector space H at 2.1.4 has dimension 3. We can now describe φ_{12}. We claim that φ_{12} is equivalent to the morphism represented by the matrix

$$\left[\begin{array}{ccc} -Y & X & 0 \\ -Z & 0 & X \\ 0 & -Y & Z \end{array}\right].$$

The argument, we recall from the proof of 3.1.3, uses the fact that the map $3\mathcal{O} \to \mathcal{F}$ is injective on global sections and the fact that the only morphism $\mathcal{O}_L(1) \to \mathcal{F}$ for any line $L \subset \mathbb{P}^2$ is the zero-morphism. We deduce that $\mathcal{C}oker(\varphi_{12})$ is isomorphic to $\mathcal{O}(1)$. We obtain $h^0(\mathcal{F}(-1)) = 1$, contradicting our hypothesis.

Assume now that C has degree 4. The zero-dimensional torsion $\mathcal{C}'$ of the quotient sheaf $\mathcal{C} = \mathcal{F}/\mathcal{O}_C$ has length at most 1, otherwise its preimage in $\mathcal{F}$ would violate stability. Assume that $\mathcal{C}'$ has length 1. Let $\mathcal{F}'$ be its preimage in $\mathcal{F}$. We have an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L \longrightarrow 0.$$

Here $L \subset \mathbb{P}^2$ is a line and it is easy to see that $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(4,-1)$. From the description of $\mathrm{M}_{\mathbb{P}^2}(4,1)$ found in [4] we know that $h^0(\mathcal{F}') \le 1$, hence $h^0(\mathcal{F}) \le 2$.

Assume, finally, that $\mathcal{C}$ has no zero-dimensional torsion. Then $\mathcal{C} \simeq \mathcal{O}_L(1)$ for a line $L \subset \mathbb{P}^2$. We have $h^0(\mathcal{F}) \le 2$ unless the morphism $\mathcal{F} \to \mathcal{O}_L(1)$ is surjective on global sections. In that case we can apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L(1) \longrightarrow 0,$$

to the standard resolution of $\mathcal{O}_C$ and, fixing an equation for L, say $X = 0$, to the resolution

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\eta} 3\mathcal{O}(-1) \xrightarrow{\xi} 2\mathcal{O} \longrightarrow \mathcal{O}_L(1) \longrightarrow 0.$$

$$\eta = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \qquad \xi = \begin{bmatrix} -Y & X & 0 \\ -Z & 0 & X \end{bmatrix}.$$

We obtain the exact sequence

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-4) \oplus 3\mathcal{O}(-1) \longrightarrow 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

We saw above that this leads to the relation $h^0(\mathcal{F}(-1)) = 1$, which is contrary to our hypothesis. $\square$

4.2. **Description of the strata as quotients.** In subsection 4.1 we found that the moduli space $\mathrm{M}_{\mathbb{P}^2}(5,0)$ can be decomposed into four strata:

- an open stratum X_0 given by the condition $h^1(\mathcal{F}) = 0$;
- a locally closed stratum X_1 of codimension 1 given by the conditions $h^0(\mathcal{F}(-1)) = 0$, $h^1(\mathcal{F}) = 1$;
- a locally closed stratum X_2 of codimension 4 given by the conditions $h^0(\mathcal{F}(-1)) = 0$, $h^1(\mathcal{F}) = 2$;
- the closed stratum X_3 given by the condition $h^0(\mathcal{F}(-1)) > 0$, consisting of sheaves of the form $\mathcal{O}_C(1)$, where $C \subset \mathbb{P}^2$ is a quintic curve. X_3 is isomorphic to $\mathbb{P}(S^5 V^*)$.

In the sequel X_i will be equipped with the canonical reduced structure induced from $\mathrm{M}_{\mathbb{P}^2}(5,0)$. Let W_0, W_1, W_2 be the sets of morphisms φ from 4.1.2, 4.1.3, respectively 4.1.4. Each sheaf $\mathcal{F}$ giving a point in X_i, $i = 0,1,2$, is the cokernel of a morphism $\varphi \in W_i$. Let $\mathbb{W}_i$ be the ambient vector spaces of homomorphisms of sheaves containing W_i, e.g. $\mathbb{W}_0 = \mathrm{Hom}(5\mathcal{O}(-2), 5\mathcal{O}(-1))$. Let G_i be the natural groups of automorphisms acting by conjugation on $\mathbb{W}_i$. In this subsection we shall prove that there exist a good quotient $W_0//G_0$, a categorical quotient of W_1 by G_1 and a geometric quotient W_2/G_2. We shall prove that each quotient is isomorphic to the corresponding subvariety X_i. We shall give concrete descriptions of $W_0//G_0$ and W_2/G_2.

Proposition 4.2.1. *There exists a good quotient $W_0//G_0$ and it is a proper open subset inside* $\mathrm{N}(3,5,5)$. *Moreover, $W_0//G_0$ is isomorphic to X_0. In particular,* $\mathrm{M}_{\mathbb{P}^2}(5,0)$ *and* $\mathrm{N}(3,5,5)$ *are birational.*

Proof. Let $\mathbb{W}_0^{ss} \subset \mathbb{W}_0$ denote the subset of morphisms that are semi-stable for the action of G_0. This group is reductive, so by the classical geometric invariant theory there is a good quotient $\mathbb{W}_0^{ss}//G_0$, which is nothing but the Kronecker moduli space $\mathrm{N}(3,5,5)$. According to King's criterion of semi-stability [7], a morphism $\varphi \in \mathbb{W}_0$ is semi-stable if and only if it is not in the G_0-orbit of a morphism of the form

$$\left[\begin{array}{cc} \star & \psi \\ \star & 0 \end{array} \right] \quad \text{for some} \quad \psi\colon (m+1)\mathcal{O}(-2) \longrightarrow m\mathcal{O}(-1), \quad 0 \le m \le 4.$$

It is now clear that W_0 is the subset of injective morphisms inside $\mathbb{W}_0^{ss}$, so it is open and G_0-invariant. In point of fact, it is easy to check that W_0 is the preimage in $\mathbb{W}_0^{ss}$ of a proper open subset inside $\mathbb{W}_0^{ss}//G_0$. This subset is the good quotient of W_0 by G_0.

We shall now prove the injectivity of the canonical map $W_0//G_0 \to X_0$. Consider the map $\upsilon\colon W_0 \to X_0$ sending φ to the stable-equivalence class of its cokernel. Consider a properly semi-stable sheaf $\mathcal{F} = \mathcal{C}oker(\varphi)$, $\varphi \in W_0$, giving a point $[\mathcal{F}]$ in X_0. For simplicity of notations we assume that $\mathcal{F}$ has a Jordan-Hölder filtration of length 2, i.e. there is an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

of stable sheaves $\mathcal{F}' \in \mathrm{M}_{\mathbb{P}^2}(r,0)$ and $\mathcal{F}'' \in \mathrm{M}_{\mathbb{P}^2}(s,0)$. From the proof of 4.1.2 we see that there are resolutions

$$0 \longrightarrow r\mathcal{O}(-2) \xrightarrow{\varphi'} r\mathcal{O}(-1) \longrightarrow \mathcal{F}' \longrightarrow 0,$$

$$0 \longrightarrow s\mathcal{O}(-2) \xrightarrow{\varphi''} s\mathcal{O}(-1) \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

Using the horseshoe lemma we see that φ is in the orbit of a morphism represented by a matrix of the form

$$\left[\begin{array}{cc} \varphi'' & 0 \\ \star & \varphi' \end{array} \right].$$

It is clear that $\varphi'' \oplus \varphi'$ is in the closure of the orbit of φ. Thus $\upsilon^{-1}([\mathcal{F}])$ is a union of orbits, each containing $\varphi'' \oplus \varphi'$ in its closure. It follows that the preimage of $[\mathcal{F}]$ in $W_0//G_0$ is a point. Thus far we have proved that the canonical map $W_0//G_0 \to X_0$ is bijective. To show that it is an isomophism we use the method of 3.1.6 [4]. We must produce resolution 4.1.2 starting

from the Beilinson spectral sequence for $\mathcal{F}$. Diagram (2.2.3) [4] for $\mathcal{F}$ reads

$$\begin{array}{ccc} 5\mathcal{O}(-2) \xrightarrow{\varphi_1} 5\mathcal{O}(-1) & & 0\,. \\ & & \\ 0 \qquad\qquad\qquad 0 & & 0 \end{array}$$

From the exact sequence (2.2.5) [4] we deduce that φ_1 is injective and its cokernel is isomorphic to $\mathcal{F}$. □

Proposition 4.2.2. *There exists a categorical quotient of W_1 modulo G_1, which is isomorphic to X_1.*

Proof. Let $\upsilon\colon W_1 \to X_1$ be the canonical map sending a morphism φ to the stable-equivalence class of its cokernel. As in the proof of 4.2.1, one can check that the preimage of an arbitrary point in X_1 under υ is a union of G_1-orbits whose closures have non-empty intersection. This shows that υ is bijective. To show that υ is a categorical quotient map we proceed as at 3.1.6 [4]. Given $\mathcal{F}$ in X_1, we need to produce resolution 4.1.3 starting from the Beilinson spectral sequence. We shall work, instead, with the dual sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$, which gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,5)$. Diagram (2.2.3) [4] for $\mathcal{G}$ takes the form

$$\begin{array}{ccc} \mathcal{O}(-2) & 0 & 0\,. \\ & & \\ \mathcal{O}(-2) \xrightarrow{\varphi_3} & 5\mathcal{O}(-1) \xrightarrow{\varphi_4} & 5\mathcal{O} \end{array}$$

The exact sequence (2.2.5) [4] reads

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{G} \longrightarrow 0.$$

Repeating the arguments from the proof of 3.2.4 it is easy to see that we may write

$$\varphi_3 = \begin{bmatrix} X \\ Y \\ Z \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \varphi_4 = \begin{bmatrix} -Y & X & 0 & \star & \star \\ -Z & 0 & X & \star & \star \\ 0 & -Z & Y & \star & \star \\ 0 & 0 & 0 & \psi_{11} & \psi_{12} \\ 0 & 0 & 0 & \psi_{21} & \psi_{22} \end{bmatrix}.$$

If the morphism $\psi\colon 2\mathcal{O}(-1) \to 2\mathcal{O}$ represented by the matrix $(\psi_{ij})_{1\le i,j\le 2}$ were not injective, then ψ would be equivalent to a morphism represented by a matrix with a zero-row or a zero-column. From the snake lemma it would follow that $\mathcal{C}oker(\varphi_4)$ has a subsheaf $\mathcal{S}$ with Hilbert polynomial $P(t) = 3t+4$ or $2t+3$. This sheaf would map injectively to $\mathcal{G}$ because $\mathcal{S} \cap \mathcal{O}(-2) = \{0\}$. The semi-stability of $\mathcal{G}$ would be violated. We deduce that ψ is injective and we obtain the extension

$$0 \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{C}oker(\psi) \longrightarrow 0,$$

which yields the resolution

$$0 \longrightarrow 2\mathcal{O}(-1) \longrightarrow 2\mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow 0.$$

Combining with the resolution of $\mathcal{G}$ from above we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow 2\mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{G} \longrightarrow 0.$$

By duality, this corresponds to resolution 4.1.3 for $\mathcal{F}$. □

Proposition 4.2.3. *There exists a geometric quotient W_2/G_2 and it is a proper open subset inside a fibre bundle over $\mathbb{P}^2 \times \mathbb{P}^2$ with fibre $\mathbb{P}^{18}$.*

Proof. The construction of W_2/G_2 is analogous to the construction of the geometric quotient W_1/G_1 from 2.2.2. Let $W_2' \subset \mathbb{W}_2$ be the locally closed subset given by the conditions $\varphi_{12} = 0$, φ_{11} has linearly independent entries, φ_{22} has linearly independent entries. The pairs of morphisms $(\varphi_{11}, \varphi_{22})$ form an open subset

$$U \subset \mathrm{Hom}(2\mathcal{O}(-3), \mathcal{O}(-2)) \times \mathrm{Hom}(\mathcal{O}(-1), 2\mathcal{O}).$$

The reductive subgroup $G_{2\mathrm{red}}$ of G_2 acts on U with kernel S and $U/(G_{2\mathrm{red}}/S)$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. Note that W_2' is the trivial bundle over U with fibre $\mathrm{Hom}(2\mathcal{O}(-3), 2\mathcal{O})$. The subset $\Sigma \subset W_2'$ given by the condition

$$\varphi_{21} = \varphi_{22}u + v\varphi_{11}, \quad u \in \mathrm{Hom}(2\mathcal{O}(-3), \mathcal{O}(-1)), \quad v \in \mathrm{Hom}(\mathcal{O}(-2), 2\mathcal{O}),$$

is a subbundle. The quotient bundle Q' has rank 19 and descends to a vector bundle Q on $U/(G_{2\mathrm{red}}/S)$ as at 2.2.2. Then $\mathbb{P}(Q)$ is the geometric quotient $(W_2' \setminus \Sigma)/G_2$.

W_2 is the open invariant subset of injective morphism inside $W_2' \setminus \Sigma$. It is a proper subset as, for instance, the morphism represented by the matrix

$$\left[\begin{array}{ccc} X & Y & 0 \\ Z^3 & 0 & Y \\ 0 & Z^3 & -X \end{array} \right]$$

is in $W_2 \setminus \Sigma$ but is not injective. We conclude that W_2/G_2 exists and is a proper open subset inside $\mathbb{P}(Q)$. □

Proposition 4.2.4. *The geometric quotient W_2/G_2 is isomorphic to X_2.*

Proof. The canonical morphism $W_2/G_2 \to X_2$ is easily seen to be injective, there being no properly semi-stable sheaves in X_2, cf. 4.1.5. To show that it is an isomorphism we must construct resolution 4.1.4 starting from the Beilinson spectral sequence of a sheaf $\mathcal{F}$ in X_2. We prefer to work, instead, with the dual sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$, which gives a point in $\mathrm{M}_{\mathbb{P}^2}(5, 5)$. Diagram (2.2.3) [4]

for $\mathcal{G}$ takes the form

$$2\mathcal{O}(-2) \xrightarrow{\varphi_1} \mathcal{O}(-1) \qquad\qquad 0 \quad .$$

$$2\mathcal{O}(-2) \xrightarrow{\varphi_3} 6\mathcal{O}(-1) \xrightarrow{\varphi_4} 5\mathcal{O}$$

As in the proof of 3.2.4, we see that $\mathcal{C}oker(\varphi_1)$ is the structure sheaf of a point $x \in \mathbb{P}^2$ and $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$. The exact sequence (2.2.5) [4] reads

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

We see from this that $\mathcal{C}oker(\varphi_4)$ has no zero-dimensional torsion. The exact sequence (2.2.4) [4] reads

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\varphi_3} 6\mathcal{O}(-1) \xrightarrow{\varphi_4} 5\mathcal{O} \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow 0.$$

We claim that φ_3 is equivalent to the morphism represented by the matrix

$$\left[\begin{array}{cccccc} X & Y & Z & 0 & 0 & 0 \\ 0 & 0 & 0 & X & Y & Z \end{array} \right]^{\mathrm{T}}.$$

Firstly, we show that any matrix representing a morphism equivalent to φ_3 has three linearly independent entries on each column. For this we use the fact that the only morphism from the structure sheaf of a point to $\mathcal{C}oker(\varphi_4)$ is the zero-morphism and we argue as in the proof that the vector space H from 2.1.4 has dimension 3. Thus φ_3 has one of the four canonical forms given in the proof of 3.2.5. Three of these can be eliminated as in the proof of 3.1.3. The argument, we recall, uses the fact that the map $5\mathcal{O} \to \mathcal{C}oker(\varphi_4)$ is injective on global sections as well as the fact that the only morphism $\mathcal{O}_L(1) \to \mathcal{C}oker(\varphi_4)$ for any line $L \subset \mathbb{P}^2$ is the zero-morphism. Indeed, such a morphism must factor through φ_5 because the composed morphism $\mathcal{O}_L(1) \to \mathcal{C}oker(\varphi_4) \to \mathcal{G}$ is zero. This follows from the fact that both $\mathcal{O}_L(1)$ and $\mathcal{G}$ are semi-stable and $p(\mathcal{O}_L(1)) > p(\mathcal{G})$.

Next we describe φ_4. Its matrix cannot be equivalent to a matrix having a zero-row. Indeed, if this were the case, then $\mathcal{C}oker(\varphi_4)$ would be isomorphic to $\mathcal{O} \oplus \mathcal{C}$, where $\mathcal{C}$ is a torsion sheaf with resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow 6\mathcal{O}(-1) \longrightarrow 4\mathcal{O} \longrightarrow \mathcal{C} \longrightarrow 0.$$

We have $P_{\mathcal{C}}(t) = 2t + 4$ and $\mathcal{C}$ maps injectively to $\mathcal{G}$ because $\mathcal{C} \cap \mathcal{O}(-3) = \{0\}$. The semi-stability of $\mathcal{G}$ is violated. We conclude that φ_4 has the form

$$\left[\begin{array}{cc} \xi & 0 \\ \star & \psi \end{array} \right],$$

where ξ is a morphism as in the proof of 4.1.6 and ψ is equivalent to the morphism φ_{12} also from 4.1.6. We have exact sequences

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow 3\mathcal{O}(-1) \xrightarrow{\xi} 2\mathcal{O} \longrightarrow \mathcal{O}_L(1) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow 3\mathcal{O}(-1) \xrightarrow{\psi} 3\mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Recall that the greatest common divisor of the maximal minors of ξ is a linear form. The line $L \subset \mathbb{P}^2$ is the zero-locus of this form. From the snake lemma we obtain an extension

$$0 \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{O}_L(1) \longrightarrow 0,$$

hence a resolution

$$0 \longrightarrow \mathcal{O} \longrightarrow 2\mathcal{O}(1) \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow 0.$$

Note that φ_5 lifts to a morphism $\mathcal{O}(-3) \to 2\mathcal{O}(1)$, so we arrive at the exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O} \longrightarrow 2\mathcal{O}(1) \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

From the horseshoe lemma we obtain a resolution

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus \mathcal{O} \longrightarrow \mathcal{O}(-1) \oplus 2\mathcal{O}(1) \longrightarrow \mathcal{G} \longrightarrow 0.$$

$\mathrm{H}^1(\mathcal{G})$ vanishes, hence $\mathcal{O}(-3)$ can be cancelled to yield the dual of resolution 4.1.4. □

4.3. **Geometric description of the strata.** Let X_0^s denote the subset of X_0 of isomorphism classes of stable sheaves. Given $\varphi \in W_0$, we write its domain $\mathcal{O}(-2) \oplus 4\mathcal{O}(-2)$ and denote by φ_{12} the restriction of φ to the second component. Let Y_0 be the open subset of X_0 of stable-equivalence classes of sheaves $\mathcal{F}$ that occur as cokernels

$$0 \longrightarrow \mathcal{O}(-2) \oplus 4\mathcal{O}(-2) \xrightarrow{\varphi} 5\mathcal{O}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$$

in which the maximal minors of φ_{12} have no common factor.

Proposition 4.3.1. *The sheaves in Y_0 have the form $\mathcal{J}_Z(3)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 10 not contained in a cubic curve, contained in a quintic curve C, and $\mathcal{J}_Z \subset \mathcal{O}_C$ is its ideal sheaf.*

The generic sheaves in X_0^s have the form $\mathcal{O}_C(3)(-P_1 - \ldots - P_{10})$, where $C \subset \mathbb{P}^2$ is a smooth quintic curve and P_i, $1 \leq i \leq 10$, are distinct points on C not contained in a cubic curve.

Proof. Consider the sheaf $\mathcal{F} = \mathcal{C}oker(\varphi)$, where the maximal minors of φ_{12} have no common factor. According to 4.5 and 4.6 [2], $\mathcal{C}oker(\varphi_{12}) \simeq \mathcal{I}_Z(3)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 10, not contained in a cubic curve. Conversely, any $\mathcal{I}_Z(3)$ is the cokernel of some morphism $\varphi_{12} \colon 4\mathcal{O}(-2) \to 5\mathcal{O}(-1)$ whose maximal minors have no common factor. It now follows, as at 2.3.4(i), that $\mathcal{F} \simeq \mathcal{J}_Z(3)$.

The claim about generic stable sheaves follows from the fact that any line bundle on a smooth curve is stable. □

Proposition 4.3.2. *The sheaves $\mathcal{F}$ in X_2 are precisely the non-split extension sheaves of the form*

$$0 \longrightarrow \mathcal{J}_x(1) \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_z \longrightarrow 0,$$

where $\mathcal{J}_x \subset \mathcal{O}_C$ is the ideal sheaf of a point x on a quintic curve $C \subset \mathbb{P}^2$ and $\mathbb{C}_z$ is the structure sheaf of a point $z \in C$. When $x = z$ we exclude the possibility $\mathcal{F} \simeq \mathcal{O}_C(1)$.

The generic sheaf in X_2 has the form $\mathcal{O}_C(1)(P - Q)$, where $C \subset \mathbb{P}^2$ is a smooth quintic curve and P, Q are distinct points on C. In particular, the closure of X_2 contains X_3.

Proof. To get the extension from the proposition we apply the snake lemma to a diagram similar to the diagram from the proof of 2.3.2. Here

$$\varphi = \left[\begin{array}{ccc} u_1 & u_2 & 0 \\ \star & \star & v_1 \\ \star & \star & v_2 \end{array} \right],$$

C is given by the equation $\det(\varphi) = 0$, x is the point given by the equations $v_1 = 0, v_2 = 0$ and z is the point given by the equations $u_1 = 0, u_2 = 0$. To prove the converse we combine the resolutions

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{I}_x(1) \longrightarrow \mathcal{J}_x(1) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \longrightarrow \mathbb{C}_z \longrightarrow 0$$

into the resolution

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{I}_x(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

If $x \neq z$, then $\operatorname{Ext}^1(\mathbb{C}_x, \mathcal{I}_x(1)) = 0$ and the arguments from the proof of 2.3.2 show that the map $\mathcal{O}(-4) \to \mathcal{O}(-4)$ in the above complex is non-zero. Canceling $\mathcal{O}(-4)$ we get the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{I}_x(1) \longrightarrow \mathcal{F} \longrightarrow 0$$

from which we immediately obtain resolution 4.1.4. A priori we have two possibilities: $h^0(\mathcal{F}) = 2$ or 3. In the first case the map $\mathcal{O}(-4) \to \mathcal{O}(-4)$ is non-zero and we are done. In the second case we can combine the resolutions

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-1) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{J}_x(1) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow 2\mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathbb{C}_z \longrightarrow 0$$

into the resolution

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-4) \oplus 3\mathcal{O}(-1) \longrightarrow 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

We saw in the proof of 4.1.6 how this resolution leads to the conclusion that $\mathcal{F}$ be isomorphic to $\mathcal{O}_C(1)$ for a quintic curve $C \subset \mathbb{P}^2$. This possibility is excluded by hypothesis.

If C is a smooth quintic curve and P converges to Q, then $\mathcal{O}_C(1)(P-Q)$ represents a point in X_2 converging to the point in X_3 represented by $\mathcal{O}_C(1)$. This shows that $X_3 \subset \overline{X}_2$. □

Proposition 4.3.3. $\{X_0, X_1, X_2, X_3\}$ *represents a stratification of* $\mathrm{M}_{\mathbb{P}^2}(5,0)$ *by locally closed irreducible subvarieties of codimensions* $0, 1, 4, 6$.

Proof. We saw above that X_3 lies in $\overline{X}_2$ and we know that X_0 is dense in $\mathrm{M}_{\mathbb{P}^2}(5,0)$. Thus we only need to show that X_2 is included in the closure of X_1. For this we shall apply the method of 3.2.3 [4]. Consider the open subset $X = \mathrm{M}_{\mathbb{P}^2}(5,0) \setminus X_3$ of stable-equivalence classes of sheaves satisfying the condition $h^0(\mathcal{F}(-1)) = 0$. Using the Beilinson monad for $\mathcal{F}(-1)$ we see that X is parametrised by an open subset M inside the space of monads of the form

$$0 \longrightarrow 10\mathcal{O}(-1) \xrightarrow{A} 15\mathcal{O} \xrightarrow{B} 5\mathcal{O}(1) \longrightarrow 0.$$

The automorphism of $\mathrm{M}_{\mathbb{P}^2}(5,0)$ taking the stable-equivalence class of a sheaf $\mathcal{F}$ to the stable-equivalence class of the dual sheaf $\mathcal{F}^{\mathrm{D}}$ leaves X invariant. Thus, in view of Serre duality, he have $h^1(\mathcal{F}(1)) = h^0(\mathcal{F}^{\mathrm{D}}(-1)) = 0$ for all $\mathcal{F}$ in X. This allows us to deduce that the map Φ defined by $\Phi(A,B) = B$ has surjective differential at every point in M. As at 3.2.3 [4], this leads to the conclusion that X_2 is included in the closure of X_1 in X, hence X_2 is included in the closure of X_1 in $\mathrm{M}_{\mathbb{P}^2}(5,0)$. □

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